Lecture 10 – Axiomatic Semantics

AAA551: Programming Language Theory

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Recall

- Continuations
  - A-Normal Form
- CC Machine
  - Recall: Evaluation Contexts and λ-Calculus
  - CC Machine
  - SCC Machine
- CK, CEK, CESK Machines
  - CK Machine
  - CEK Machine
  - CESK Machine
  - Variants of CEK Machine

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Semantics

Three approaches to **formal semantics**:

- **Operational** $\sigma \vdash e \Rightarrow v$
  - *How* is a program executed?
  - Useful for implementation of compilers and interpreters.

$$\frac{\vdash e_1 \Rightarrow n_1 \qquad \vdash e_2 \Rightarrow n_2}{\vdash e_1 \texttt{ + } e_2 \Rightarrow n_1 + n_2}$$

- **Denotational** $[\![e]\!]$
  - *What* is the mathematical object for a program?
  - Useful for theoretical foundations.

$$[\![e_1 \texttt{ + } e_2]\!] = [\![e_1]\!] + [\![e_2]\!]$$

- **Axiomatic** $\vdash \{\phi\}\ e\ \{\phi'\}$
  - *Which properties* does a program satisfy?
  - Useful for proving program properties and correctness.

$$\vdash \{x < n \wedge y < m\}\ z \texttt{ = } x \texttt{ + } y\ \{z < n + m\}$$

Hoare Triples

In axiomatic semantics, a **Hoare triple** represents properties of a statement s:

$$\{\phi\}\ s\ \{\phi'\}$$

which means that if the statement s is executed in a state satisfying the **precondition** ϕ and **terminates**, then the **postcondition** ϕ' will hold in the resulting state.

For example,

$$\{x < n \land y < m\}\ z = x + y\ \{z < n + m\}$$

$$\{0 \leq n \land i = n \land x = 1\}$$

```
while (0 < i) {
    x = x * i;
    i = i - 1;
}
```

$$\{x = n!\}$$

Partial Correctness vs. Total Correctness

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There are two types of correctness properties:

- **Partial correctness**: **If** a statement s is executed in a state satisfying ϕ and **terminates**, then ϕ' will hold in the resulting state:

$$\{\phi\}\ s\ \{\phi'\}$$

- **Total correctness**: If a statement s is executed in a state satisfying ϕ, **then** it **terminates** and ϕ' will hold in the resulting state:

$$[\phi]\ s\ [\phi']$$

The main difference is that total correctness requires termination, while partial correctness does not.

Partial Correctness vs. Total Correctness

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Which ones are correct?

1. {false} `while (true) {}` {false}
2. {true} `while (true) {}` {false}
3. [false] `while (true) {}` [false]
4. [true] `while (true) {}` [false]

The 1 and 3 are **correct** because no state satisfies the precondition.

The 2 is **correct** because it does not terminate, so the postcondition is never violated even though it is `false`.

The 4 is **incorrect** because it does not terminate, which violates the total correctness requirement.

Assertion Language

Recall the language IMP we defined in previous lectures.

$$
\begin{array}{lll}
\text{Expressions} & e & ::= n \mid x \mid e \texttt{ + } e \mid e \texttt{ * } e \mid e \texttt{ < } e \mid \texttt{true} \mid \texttt{false} \\
\text{Statements} & s & ::= \texttt{skip} \mid x \texttt{ := } e \mid s\texttt{;}s \\
 & & \quad \mid \texttt{if } e \texttt{ then } s \texttt{ else } s \\
 & & \quad \mid \texttt{while } e \texttt{ do } s \\
\text{Values} & v & ::= n \mid \texttt{true} \mid \texttt{false}
\end{array}
$$

Let's define an **assertion language** for IMP.

$$
\begin{array}{lll}
i, j & \in & \textbf{LVar} \\
a & ::= & x \mid i \mid n \mid a \texttt{ + } a \mid a \texttt{ * } a \\
\phi & ::= & \texttt{true} \mid \texttt{false} \mid a \texttt{ = } a \mid a \texttt{ < } a \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x.\phi \mid \forall x.\phi
\end{array}
$$

where **LVar** is a set of **logical variables** different from the program variables in $\mathbb{X}$ (e.g., x, y, z), a is an **arithmetic expression** used in assertions, and ϕ is a **assertion** that can be true or false.

Denotational Semantics of Arithmetic Expressions

A **program state** $\sigma : \mathbb{X} \to \mathbb{V}$ is a mapping from program variables to values (ignoring the labels $\mathbb{L}$ for now).

In addition, we need an **interpretation** I for logical variables, which is a mapping from logical variables to integers:

$$I : \textbf{LVar} \to \mathbb{Z}$$

We can define the **denotational semantics** of **arithmetic expressions** with a given pair of state σ and interpretation I:

$$\boxed{\mathcal{A}[\![a]\!](\sigma, I) : \mathbb{Z}}$$

$$\begin{array}{lcll}
\mathcal{A}[\![x]\!](\sigma, I) & = & \sigma(x) & \text{only if } \sigma(x) \in \mathbb{Z} \\
\mathcal{A}[\![i]\!](\sigma, I) & = & I(i) & \\
\mathcal{A}[\![n]\!](\sigma, I) & = & n & \\
\mathcal{A}[\![a_1 \texttt{ + } a_2]\!](\sigma, I) & = & \mathcal{A}[\![a_1]\!](\sigma, I) + \mathcal{A}[\![a_2]\!](\sigma, I) & \\
\mathcal{A}[\![a_1 \texttt{ * } a_2]\!](\sigma, I) & = & \mathcal{A}[\![a_1]\!](\sigma, I) \times \mathcal{A}[\![a_2]\!](\sigma, I) &
\end{array}$$

Satisfaction

Now, we can define the **satisfaction relation** $\models_I$ for assertions:

$$\boxed{\sigma \models_I \phi}$$

$$
\begin{array}{ll}
\sigma \models_I \texttt{true} & \\
\sigma \models_I a_1 = a_2 & \text{if } \mathcal{A}[\![a_1]\!](\sigma, I) = \mathcal{A}[\![a_2]\!](\sigma, I) \\
\sigma \models_I a_1 < a_2 & \text{if } \mathcal{A}[\![a_1]\!](\sigma, I) < \mathcal{A}[\![a_2]\!](\sigma, I) \\
\sigma \models_I \neg\phi & \text{if } \neg\sigma \models_I \phi \\
\sigma \models_I \phi_1 \wedge \phi_2 & \text{if } \sigma \models_I \phi_1 \wedge \sigma \models_I \phi_2 \\
\sigma \models_I \phi_1 \vee \phi_2 & \text{if } \sigma \models_I \phi_1 \vee \sigma \models_I \phi_2 \\
\sigma \models_I \exists i.\phi & \text{if } \exists k \in \mathbb{Z}.\ \sigma \models_{I[i \mapsto k]} \phi \\
\sigma \models_I \forall i.\phi & \text{if } \forall k \in \mathbb{Z}, \sigma \models_{I[i \mapsto k]} \phi
\end{array}
$$

Satisfaction

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Definition (Partial Correctness Statement Satisfiability)

A partial correctness statement $\{\phi\}\ s\ \{\phi'\}$ is **satisfied** in a state σ and an interpretation I:

$$\sigma \models_I \{\phi\}\ s\ \{\phi'\}$$

if and only if the following condition holds:

$$\sigma \models_I \phi \land S[\![s]\!](\sigma) = \sigma' \implies \sigma' \models_I \phi'$$

Note that $S[\![s]\!](\sigma) = \sigma'$ means that the execution of statement s from state σ terminates in state σ'.

The definition of partial correctness statement satisfiability does not require termination.

Validity

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Definition (Assertion Validity)

An assertion ϕ is **valid** if and only if it holds for all states and interpretations:

$$\models \phi \iff \forall \sigma, I.\ \sigma \models_I \phi$$

Definition (Partial Correctness Statement Validity)

A partial correctness statement $\{\phi\}\ s\ \{\phi'\}$ is **valid** if and only if it is satisfied in all states and interpretations:

$$\models \{\phi\}\ s\ \{\phi'\} \iff \forall \sigma, I.\ \sigma \models_I \{\phi\}\ s\ \{\phi'\}$$

Then, how to prove the validity of a partial correctness statement?

We can use **Hoare logic**, which provides a set of inference rules to derive valid partial correctness statements.

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Hoare Logic

Let's define the inference rules for Hoare logic, which allow us to derive valid partial correctness statements.

We will define the following **judgment**:

$$\vdash \{\phi\}\ s\ \{\phi'\}$$

which means that the partial correctness statement $\{\phi\}\ s\ \{\phi'\}$ is derivable using the inference rules of Hoare logic.

Hoare Logic – Skip

$$\textsc{Skip}\ \frac{}{\vdash \{\phi\}\ \texttt{skip}\ \{\phi\}}$$

The precondition and postcondition are the same for the `skip` statement.

For example,

$$\textsc{Skip}\ \frac{}{\vdash \{x < n\}\ \texttt{skip}\ \{x < n\}}$$

Hoare Logic – Assignment

$$\textsc{Assign}\ \frac{}{\vdash \{\phi[x \mapsto a]\}\ x\ \texttt{:=}\ a\ \{\phi\}}$$

To satisfy the postcondition ϕ after the assignment x := a, the precondition must be ϕ with x replaced by a (i.e., $\phi[x \mapsto a]$).

For example,

$$\textsc{Assign}\ \frac{}{\vdash \{\texttt{true}\}\ x\ \texttt{:=}\ 42\ \{x = 42\}}$$

$$\textsc{Assign}\ \frac{}{\vdash \{x < n\}\ x\ \texttt{:=}\ x + 1\ \{x < n + 1\}}$$

Hoare Logic – Assignment (Incorrect)

The following inference rule for assignment is **incorrect**:

$$\textsc{Assign}\ \frac{}{\vdash \{\phi\}\ x\ \texttt{:=}\ a\ \{\phi[x \mapsto a]\}}$$

For example,

$$\textsc{Assign}\ \frac{}{\vdash \{x = 0\}\ x\ \texttt{:=}\ 5\ \{5 = 0\}}$$

Another **incorrect** inference rule for assignment is:

$$\textsc{Assign}\ \frac{}{\vdash \{\phi\}\ x\ \texttt{:=}\ a\ \{\phi[a \mapsto x]\}}$$

For example,

$$\textsc{Assign}\ \frac{}{\vdash \{x = 0\}\ x\ \texttt{:=}\ 5\ \{x = 0\}}$$

Hoare Logic – Sequence

$$\textsc{Seq}\ \frac{\vdash \{\phi\}\ s_1\ \{\phi''\} \qquad \vdash \{\phi''\}\ s_2\ \{\phi'\}}{\vdash \{\phi\}\ s_1 ; s_2\ \{\phi'\}}$$

To satisfy the postcondition ϕ' after executing $s_1 ; s_2$, we need to find an intermediate assertion ϕ'' such that s_1 satisfies ϕ'' and s_2 satisfies ϕ'.

For example,

$$\textsc{Seq}\ \frac{\vdash \{\texttt{true}\}\ x\ \texttt{:=}\ 42\ \{x = 42\} \qquad \vdash \{x = 42\}\ y\ \texttt{:=}\ x + 1\ \{y = 43\}}{\vdash \{\texttt{true}\}\ x\ \texttt{:=}\ 42;\ y\ \texttt{:=}\ x + 1\ \{y = 43\}}$$

Hoare Logic – Conditionals

$$\textsc{If}\ \frac{\vdash \{\phi \wedge e\}\ s_1\ \{\phi'\} \qquad \vdash \{\phi \wedge \neg e\}\ s_2\ \{\phi'\}}{\vdash \{\phi\}\ \texttt{if}\ e\ \texttt{then}\ s_1\ \texttt{else}\ s_2\ \{\phi'\}}$$

To satisfy the postcondition ϕ' after executing the conditional statement, we need to ensure that

- s_1 satisfies ϕ' when the condition e is true, and
- s_2 satisfies ϕ' when the condition e is false.

For example,

$$\textsc{If}\ \frac{\vdash \{x > 0\}\ y\ \texttt{:=}\ x\ \{y \geq 0\} \qquad \vdash \{x \leq 0\}\ y\ \texttt{:=}\ -1\ \texttt{*}\ x\ \{y \geq 0\}}{\vdash \{\texttt{true}\}\ \texttt{if}\ x > 0\ \texttt{then}\ y\ \texttt{:=}\ x\ \texttt{else}\ y\ \texttt{:=}\ -1\ \texttt{*}\ x\ \{y \geq 0\}}$$

Hoare Logic – Loops

$$\textsc{While}\ \frac{\vdash \{\phi \wedge e\}\ s\ \{\phi\}}{\vdash \{\phi\}\ \texttt{while}\ e\ \texttt{do}\ s\ \{\phi \wedge \neg e\}}$$

The precondition ϕ is called the **loop invariant**, which must hold before and after each iteration of the loop. The postcondition $\phi \wedge \neg e$ holds when the loop terminates.

For example,

$$\textsc{While}\ \frac{\vdash \{i < n\}\ i\ \texttt{:=}\ i\ \texttt{+}\ 1\ \{i \leq n\}}{\vdash \{i \leq n\}\ \texttt{while}\ i\ \texttt{<}\ n\ \texttt{do}\ i\ \texttt{:=}\ i\ \texttt{+}\ 1\ \{i = n\}}$$

Hoare Logic – Consequences

$$\textsc{Consequence}\ \frac{\vdash \phi \Rightarrow \phi_1 \qquad \vdash \{\phi_1\}\ s\ \{\phi_1'\} \qquad \vdash \phi_1' \Rightarrow \phi'}{\vdash \{\phi\}\ s\ \{\phi'\}}$$

We can always **strengthen precondition** or **weaken postcondition** of a valid partial correctness statement.

For example,

$$\textsc{Consequence}\ \frac{\begin{array}{c}\vdash x < n \Rightarrow x < n+1 \\ \vdash \{x < n+1\}\ x := x+1\ \{x < n+2\} \\ \vdash x < n+2 \Rightarrow x < n+3\end{array}}{\vdash \{x < n\}\ x := x+1\ \{x < n+3\}}$$

Example – Factorial

$\{0 \leq n \wedge i = n \wedge x = 1\}$

$\{0 \leq n \wedge i = n \wedge x \times i! = n!\}$

$\{0 \leq n \wedge 0 \leq i \leq n \wedge x \times i! = n!\}$

```
while (0 < i) {
```

$\{0 \leq n \wedge 0 < i \leq n \wedge x \times i! = n!\}$

```
    x = x * i;
```

$\{0 \leq n \wedge 0 < i \leq n \wedge x \times (i-1)! = n!\}$

```
    i = i - 1;
```

$\{0 \leq n \wedge 0 \leq i < n \wedge x \times i! = n!\}$

$\{0 \leq n \wedge 0 \leq i \leq n \wedge x \times i! = n!\}$

```
}
```

$\{0 \leq n \wedge i = 0 \wedge x \times i! = n!\}$

$\{x = n!\}$

Soundness and Completeness

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Theorem (Soundness)

All derivable partial correctness statements are valid:

$$\vdash \{\phi\}\ s\ \{\phi'\} \implies \models \{\phi\}\ s\ \{\phi'\}$$

We can prove the soundness of Hoare logic by induction.

Theorem (Completeness)

All valid partial correctness statements are derivable in Hoare logic:

$$\models \{\phi\}\ s\ \{\phi'\} \implies \vdash \{\phi\}\ s\ \{\phi'\}$$

The following statement is valid only if s does not terminate.

$$\{\texttt{true}\}\ s\ \{\texttt{false}\}$$

If Hoare logic is complete, $\vdash \{\texttt{true}\}\ s\ \{\texttt{false}\}$ must be derivable.

However, it is impossible to solve the halting problem, which means that Hoare logic is **not complete** for all statements.

Relative Completeness

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Theorem (Relative Completeness)

If we assume that all valid assertions are provable in the assertion logic, then all valid partial correctness statements are derivable in Hoare logic:

$$(\forall \phi.\ \models \phi \implies \vdash \phi) \implies (\forall \phi, s, \phi'.\ \models \{\phi\}\ s\ \{\phi'\} \implies \vdash \{\phi\}\ s\ \{\phi'\})$$

So, if we have an **oracle** that can prove all valid assertions, we can use it to derive all valid partial correctness statements in Hoare logic.

If we do not have a derivation (proof) of a valid partial correctness statement, the reason must be that we cannot prove some valid assertion used in the proof, not because of the incompleteness of Hoare logic itself.

Next Lecture

- Systematic Program Proofs

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