Lecture 14 – Type System

AAA551: Programming Language Theory

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2026 Spring

Recall

- Operational semantics
  - Big-step operational semantics
  - Small-step operational semantics
  - Evaluation contexts and Lambda calculus
- Denotational semantics
  - Fixed-point theory
  - Trace Semantics
  - Trace Properties
  - Abstract Machines
- Axiomatic semantics
  - Hoare Logic
  - Systematic program proofs
  - Separation Logic

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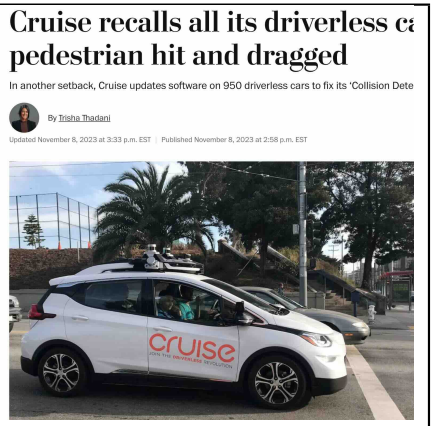
Contents

Errors in Saftety-Critical Software

Unexpected errors in **safety-critical software** cause serious problems: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			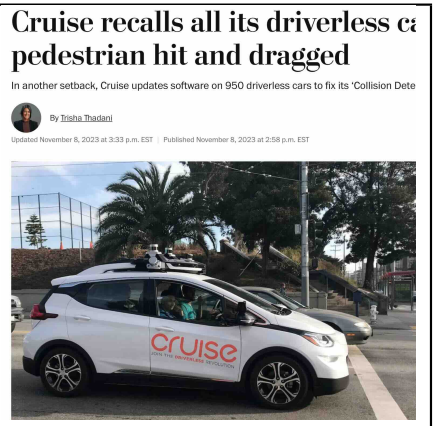
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			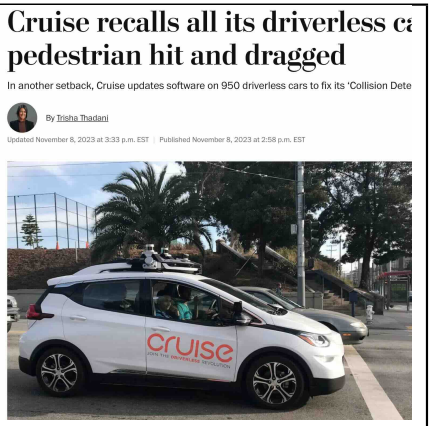
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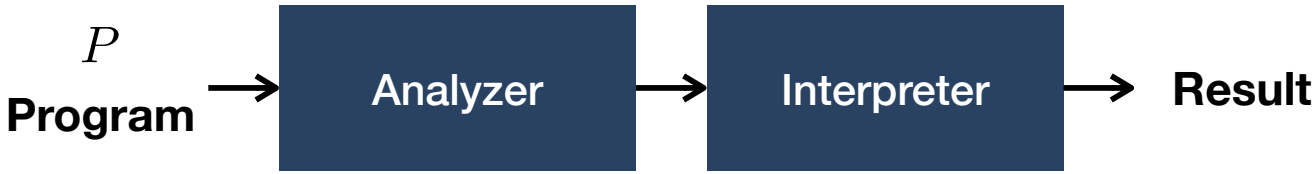
Then, how can we **prevent** such errors?

Can we **automatically** check whether a program does not have any **run-time errors**?

Detecting Run-Time Errors

We can use various **analysis** techniques to detect run-time errors:



An **analyzer** is a program that takes a program as an input and determines whether the program has a certain property. In this case, the property is **run-time errors**.

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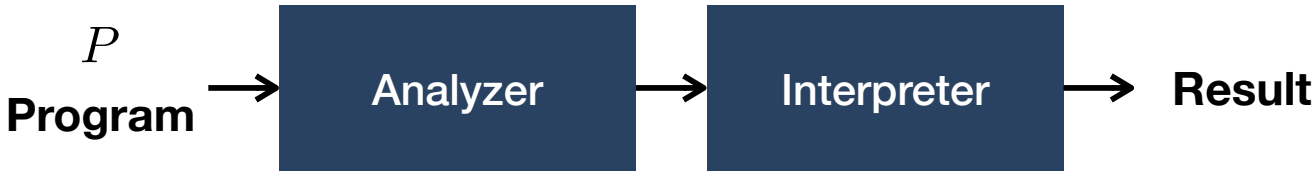

An **analyzer** is a program that takes a program as an input and determines whether the program has a certain property. In this case, the property is **run-time errors**.

We can categorize them into two groups:

- **Dynamic Analysis**: analyze programs by **executing** them
- **Static Analysis**: analyze programs **without executing** them

Dynamic Analysis

Dynamic analysis is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
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L1	-5
L2	-5
L3	5
L4	
L5	
L6	5

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L1	-5	42
L2	-5	
L3	5	
L4		42
L5		42
L6	5	42

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L1	-5	42	-7	99	0	...
L2	-5		-7			...
L3	5		7			...
L4		42		99	0	...
L5		42		99	0	...
L6	5	42	7	99	0	...

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We can easily get the **behavior** of the program for each **single input**.

However, it is **difficult** to get all the **possible behaviors** of the program for **all the inputs**.

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Let's perform **static analysis** for the following Scala program:

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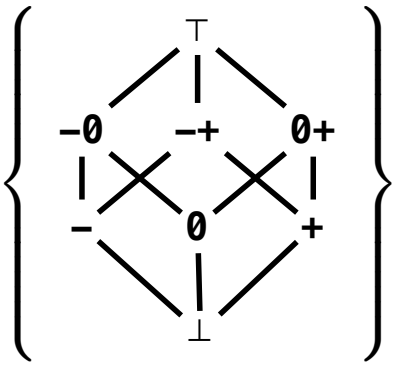
$\mathbb{D} =$ 

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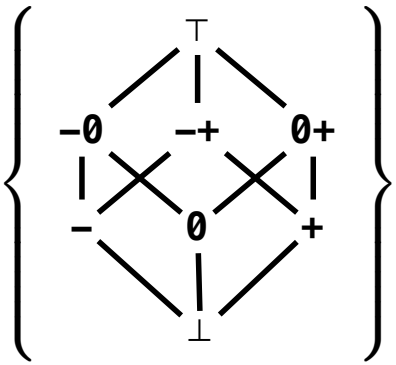
$\mathbb{D} =$ 

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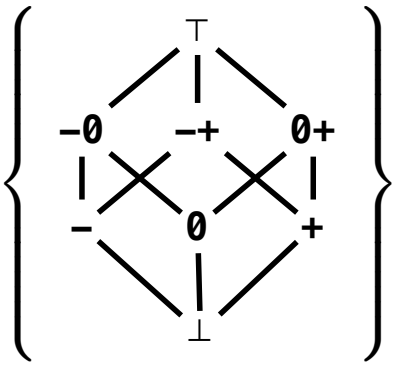
$\mathbb{D} =$ 

L1	
L2	
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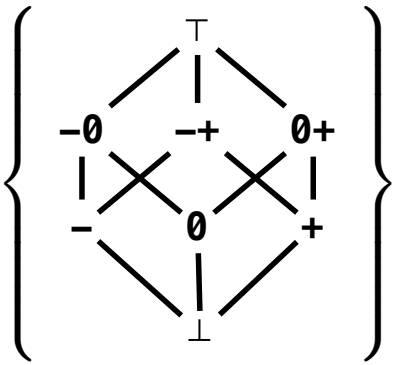
$\mathbb{D} =$ 

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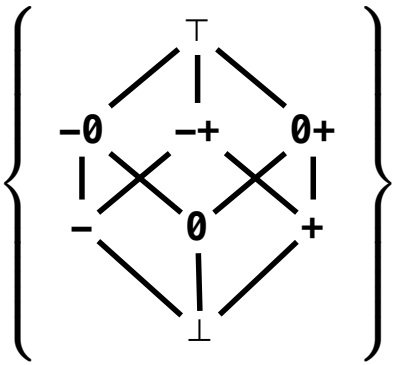
$\mathbb{D} =$ 

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We can prove that `abs` always returns a **non-negative** integer (i.e., 0+).

Soundness vs Completeness

- $\vdash \psi$ denotes that a statement ψ is **provable**.
- $\models \psi$ denotes that a statement ψ is **true**.

In a **sound** proof system, all **provable** statements are **true**.

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In a **complete** proof system, all **true** statements are **provable**.

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Analysis techniques can be used to prove that a program is **error-free**.

- $\vdash P$ denotes that a program P is **analyzed** as error-free.
- $\models P$ denotes that a program P is truly **error-free**.

Then, is dynamic/static analysis **sound** or **complete**?

Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
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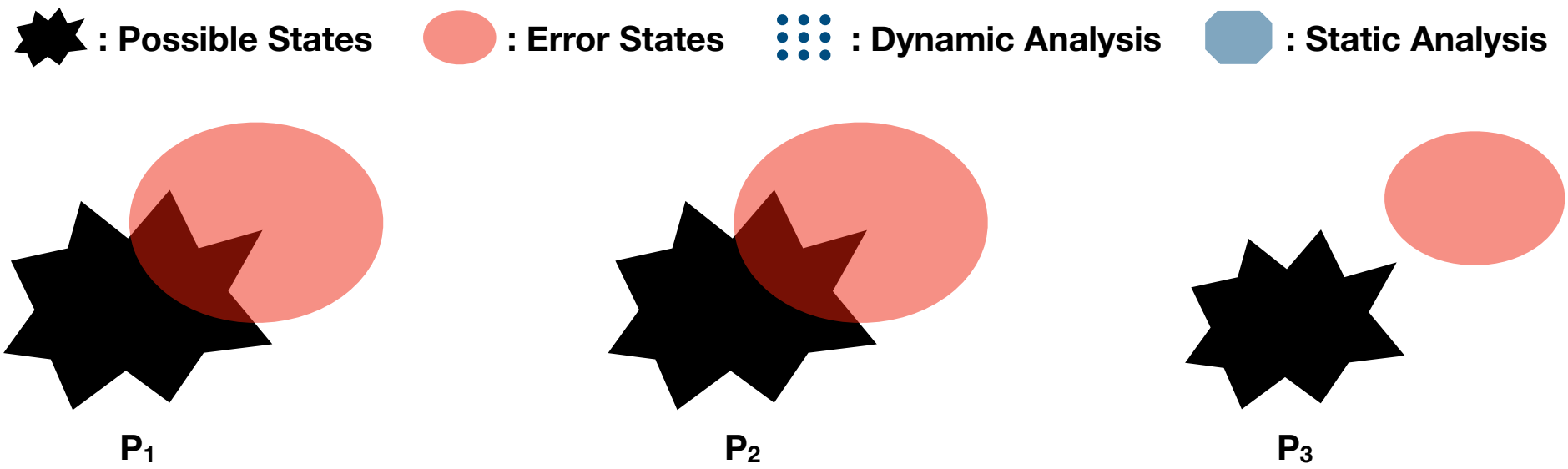

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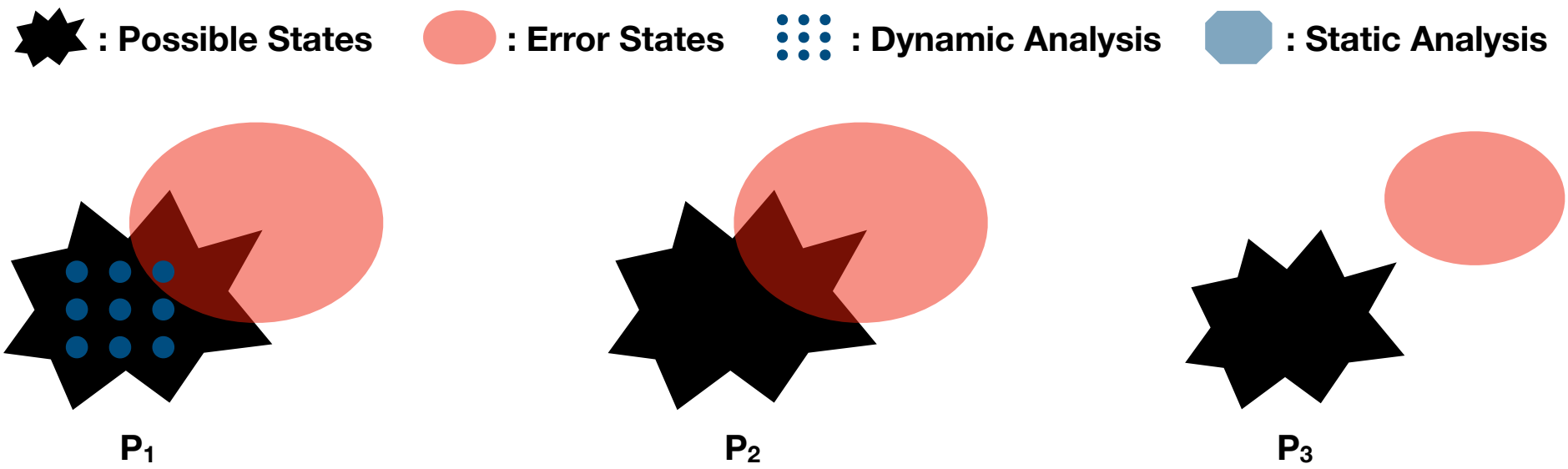

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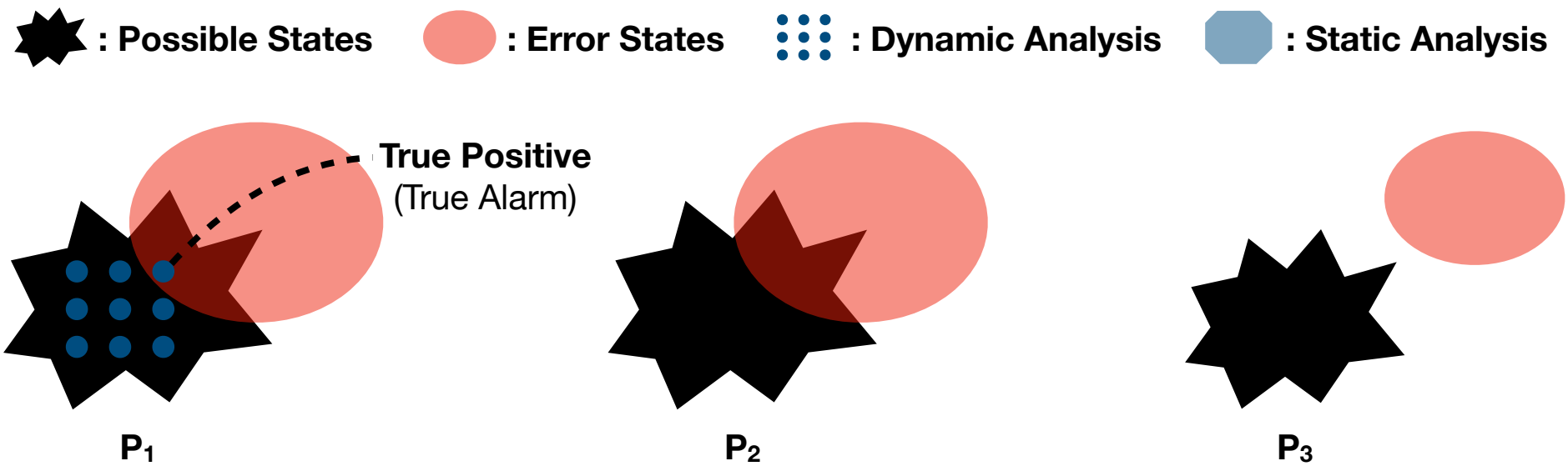

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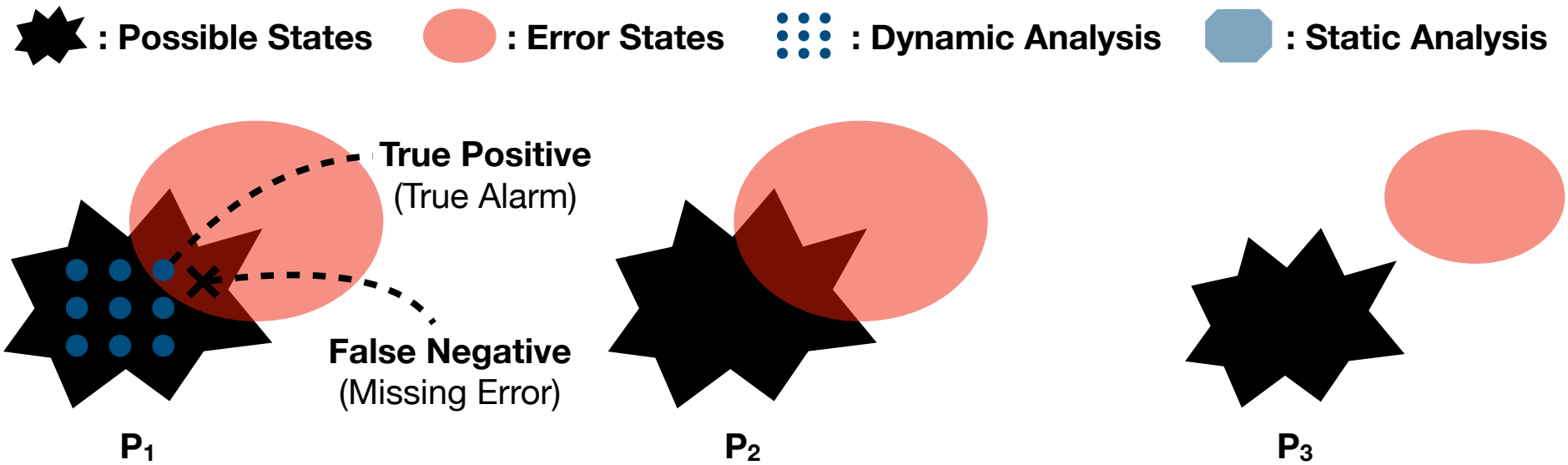

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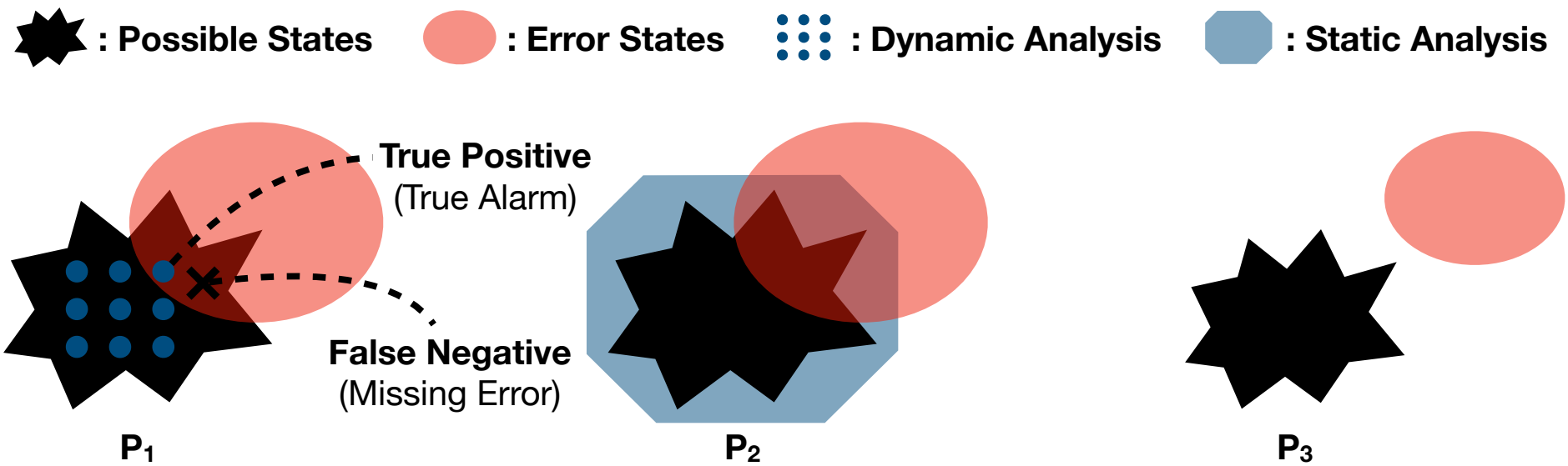

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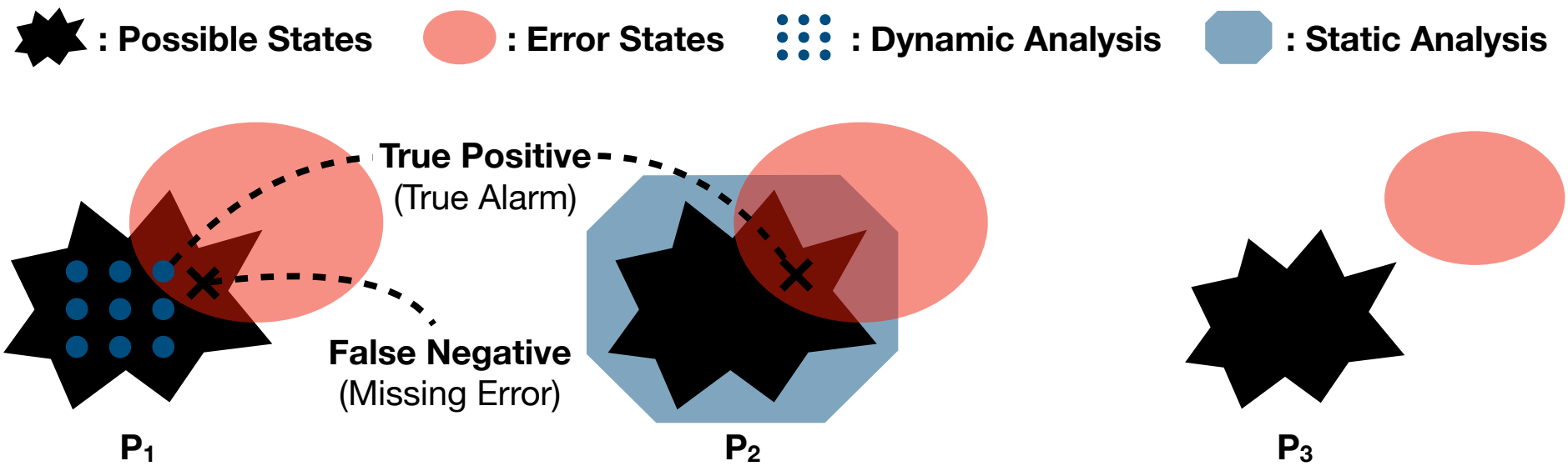

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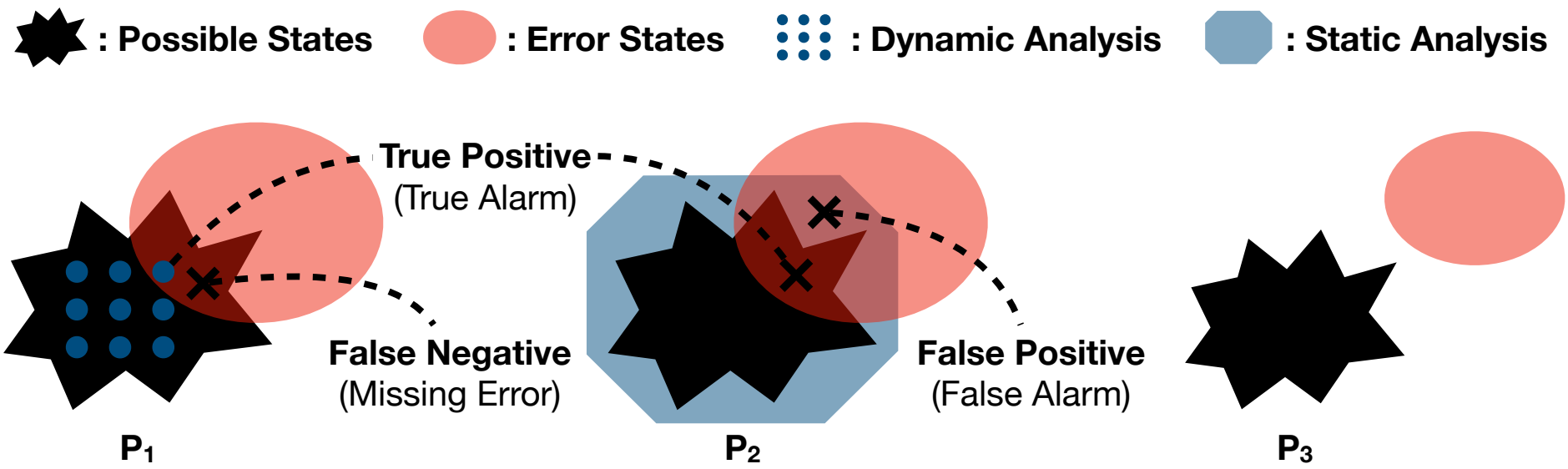

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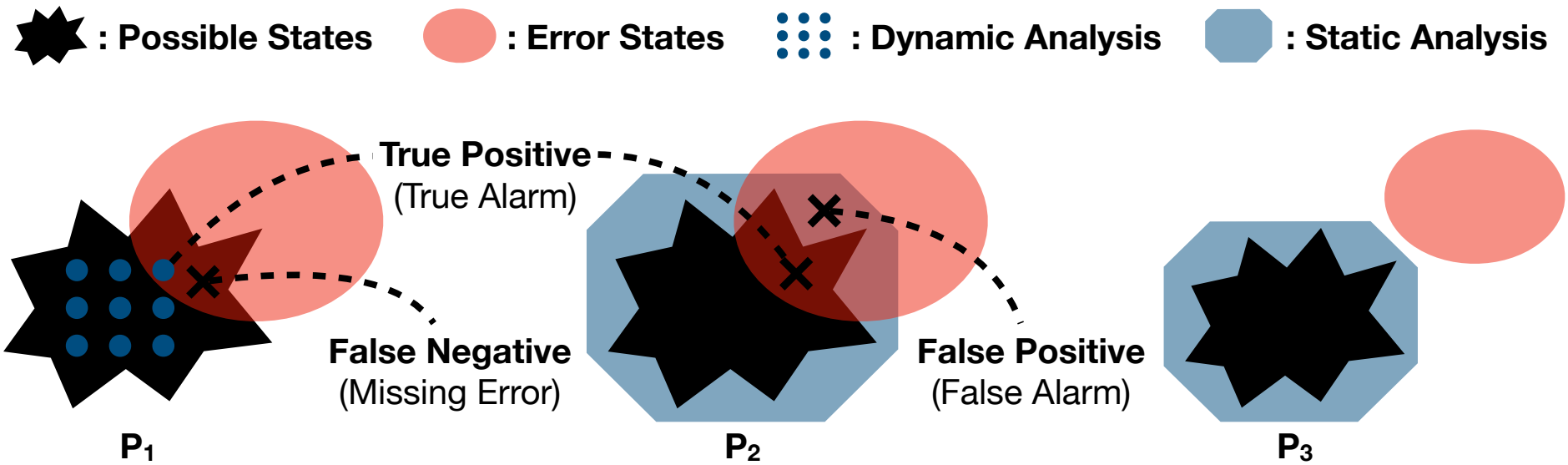

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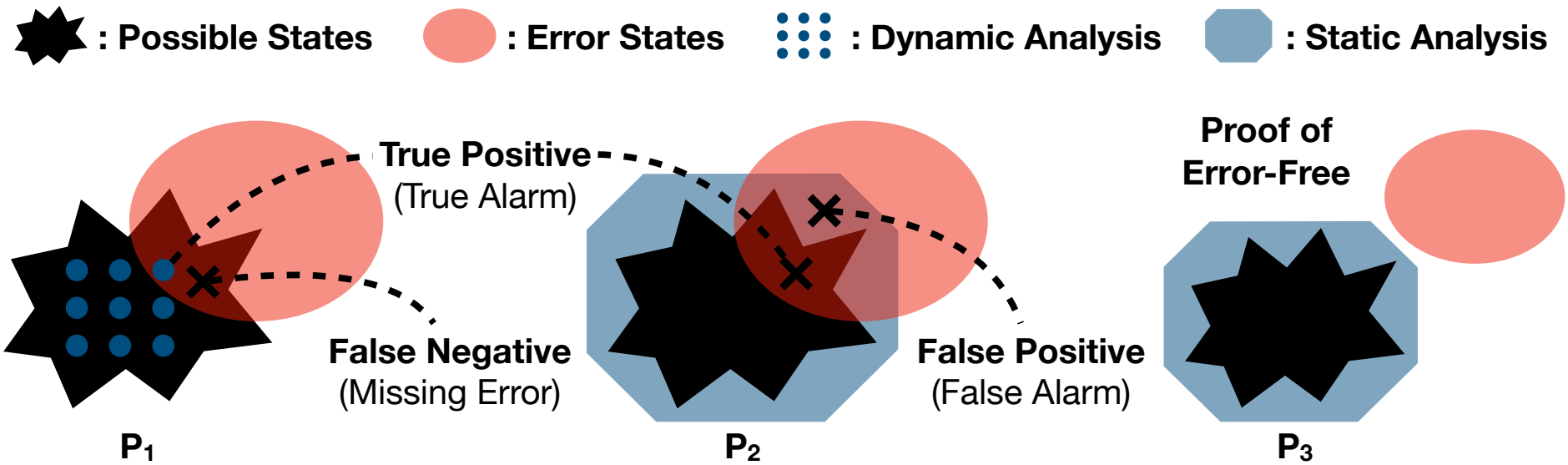

Contents

Types

PLRG

Definition (Types)

A **type** is a set of values.

For example, the `Int`, `Boolean`, and `Int => Int` types are defined as the following sets of values in Scala.

$$
\begin{aligned}
\texttt{Int} &= \{n \in \mathbb{Z} \mid -2^{31} \leq n < 2^{31}\} \\
\texttt{Boolean} &= \{\texttt{true}, \texttt{false}\} \\
\texttt{Int => Int} &= \{f \mid f \text{ is a function from } \texttt{Int} \text{ to } \texttt{Int}\}
\end{aligned}
$$

```
val n: Int = 42                 // 42    : Int
n + 1                           // 43    : Int
val b: Boolean = n > 10         // true  : Boolean
def f(x: Int): Int = x + 1      // f     : Int => Int
f(42)                           // 43    : Int
```

Type Errors

PLRG

Definition (Type Errors)

A **type error** occurs when a program tries to use a value having a type that is **incompatible** with the expected type.

For example, the following Scala program has type errors:

```
42 + true              // `Int` expected for `+`, but `Boolean` found
if (1) 2 else 3        // `Boolean` expected for `if`, but `Int` found
def f(x: Int): Int = x + 1
f(false)               // `Int` expected for `f`, but `Boolean` found
```

However, not all **run-time errors** are **type errors**:

```
42 / 0                 // `ArithmeticException` at run-time
case class A(k: Int)
val x: A = null
x.k                    // `NullPointerException` at run-time
```

Type Checking

If the following conditions hold, we say **"the expression e has type τ"**:

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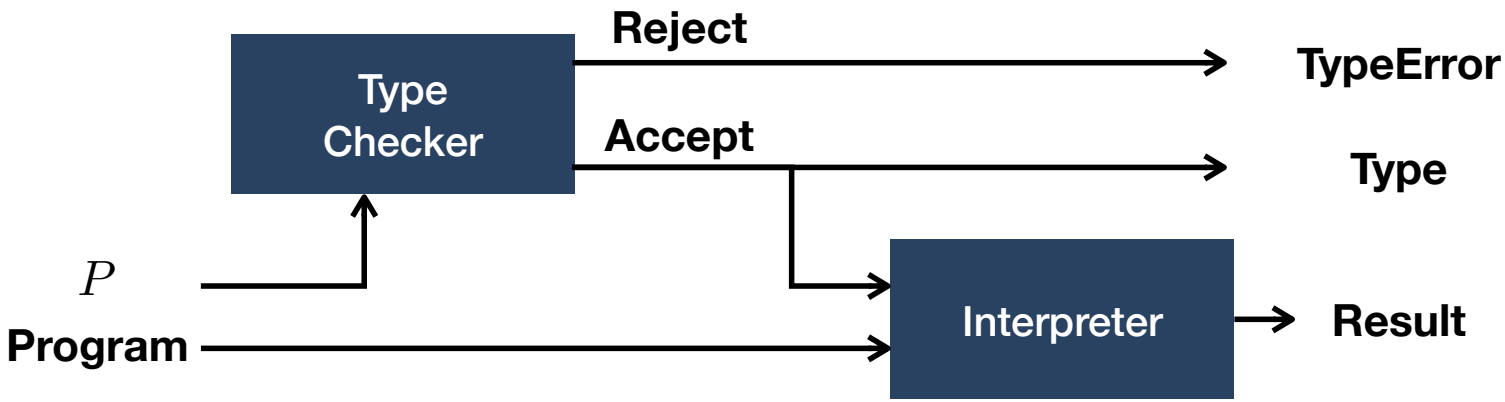
Definition (Type Checking)

Type checking is a kind of static analysis checking whether a given expression e is **well-typed**. A **type checker** returns the **type** of e if it is well-typed, or rejects it and reports the detected **type error** otherwise.



Type Soundness

PLRG

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A **type system** is **sound** if it guarantees that a **well-typed** program will **never** cause a **type error** at run-time.

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Type systems in most statically-typed languages are designed to be **sound**.

Next Lecture

- Simply-Typed Lambda Calculus

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