Lecture 3 – Small-Step Operational Semantics

AAA551: Programming Language Theory

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2026 Spring

Recall

- Programming languages
- Mathematical preliminaries
  - Binary relations and functions
  - Inductively defined sets
- Big-step operational semantics
  - Example: VAE
  - Scala implementation
  - Derivation
- Simple imperative language – IMP
  - Big-step operational semantics of IMP
  - Equivalence of statements
  - Determinism

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Operational Semantics

There are two main styles of operational semantics:

- **Big-Step Operational Semantics** defines the meaning of a program by specifying how it executes on a machine in **one big step**.

$$\frac{\cdots}{\vdash e \Rightarrow n}$$

(An expression e evaluates to n.)

- **Small-Step Operational Semantics** defines the meaning of a program by specifying how it executes on a machine **step-by-step**.

$$e \to e' \to e'' \to \ldots \to n$$

(An expression e is reduced to e', then to e'', and so on until n.)

Big-Step Operational Semantics – VAE

$$e ::= n \mid e \texttt{ + } e \mid e \texttt{ * } e \mid x \texttt{ := } e; \; e \mid x$$

$$\boxed{\sigma \vdash e \Rightarrow n}$$

$$\frac{}{\sigma \vdash n \Rightarrow n}$$

$$\frac{\sigma \vdash e_1 \Rightarrow n_1 \qquad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 \texttt{ + } e_2 \Rightarrow n_1 + n_2} \qquad \frac{\sigma \vdash e_1 \Rightarrow n_1 \qquad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 \texttt{ * } e_2 \Rightarrow n_1 \times n_2}$$

$$\frac{\sigma \vdash e_1 \Rightarrow n_1 \qquad \sigma[x \mapsto n_1] \vdash e_2 \Rightarrow n_2}{\sigma \vdash x \texttt{ := } e_1; \; e_2 \Rightarrow n_2} \qquad \frac{x \in \mathsf{dom}(\sigma)}{\sigma \vdash x \Rightarrow \sigma(x)}$$

Small-Step Operational Semantics – VAE

The small-step operational semantics of VAE is defined by the following **small-step evaluation relation** $\rightarrow$ on configurations $\langle \sigma, e \rangle$:

$$\boxed{\langle \sigma, e \rangle \rightarrow \langle \sigma, e \rangle}$$

where a **configuration** $\langle \sigma, e \rangle$ consists of

1. an environment $\sigma : \mathbb{X} \rightharpoonup \mathbb{Z}$ that maps variables to integers, and
2. an expression e.

Small-Step Operational Semantics – VAE

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$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 \texttt{ + } e_2 \rangle \to \langle \sigma', e_1' \texttt{ + } e_2 \rangle}$$

$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 \texttt{ * } e_2 \rangle \to \langle \sigma', e_1' \texttt{ * } e_2 \rangle}$$

$$\frac{\langle \sigma, e_2 \rangle \to \langle \sigma', e_2' \rangle}{\langle \sigma, n_1 \texttt{ + } e_2 \rangle \to \langle \sigma', n_1 \texttt{ + } e_2' \rangle}$$

$$\frac{\langle \sigma, e_2 \rangle \to \langle \sigma', e_2' \rangle}{\langle \sigma, n_1 \texttt{ * } e_2 \rangle \to \langle \sigma', n_1 \texttt{ * } e_2' \rangle}$$

$$\frac{}{\langle \sigma, n_1 \texttt{ + } n_2 \rangle \to \langle \sigma, n_1 + n_2 \rangle}$$

$$\frac{}{\langle \sigma, n_1 \texttt{ * } n_2 \rangle \to \langle \sigma, n_1 \times n_2 \rangle}$$

Small-Step Operational Semantics – VAE

$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, x := e_1;\ e_2 \rangle \to \langle \sigma', x := e_1';\ e_2 \rangle}$$

$$\frac{}{\langle \sigma, x := n;\ e_2 \rangle \to \langle \sigma[x \mapsto n], e_2 \rangle}$$

$$\frac{x \in \mathsf{dom}(\sigma)}{\langle \sigma, x \rangle \to \langle \sigma, \sigma(x) \rangle}$$

Small-Step Operational Semantics – VAE

We can define the **multi-step evaluation** relation $\rightarrow^*$ as the reflexive and transitive closure of $\rightarrow$:

$$\frac{}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma, e \rangle}$$

$$\frac{\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle \qquad \langle \sigma', e' \rangle \rightarrow \langle \sigma'', e'' \rangle}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle}$$

Derivation

$$
\begin{aligned}
\langle \varnothing, \mathtt{x := 1;\ \{y := 2;\ x + y\}} \rangle &\rightarrow \langle \sigma_0, \mathtt{y := 2;\ x + y} \rangle \\
&\rightarrow \langle \sigma_1, \mathtt{x + y} \rangle \\
&\rightarrow \langle \sigma_1, 1 + \mathtt{y} \rangle \\
&\rightarrow \langle \sigma_1, 1 + 2 \rangle \\
&\rightarrow \langle \sigma_1, 3 \rangle
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_0 &= [\mathtt{x} \mapsto 1] \\
\sigma_1 &= [\mathtt{x} \mapsto 1, \mathtt{y} \mapsto 2]
\end{aligned}
$$

Derivation

$$\begin{aligned}\langle \varnothing, \texttt{x := 1; } \{\texttt{y := 2; x + y}\}\rangle &\to \langle \sigma_0, \texttt{y := 2; x + y}\rangle \\ &\to \langle \sigma_1, \texttt{x + y}\rangle \\ &\to \langle \sigma_1, 1 \texttt{ + y}\rangle \\ &\to \langle \sigma_1, 1 \texttt{ + } 2\rangle \\ &\to \langle \sigma_1, 3\rangle\end{aligned}$$

where

$$\begin{aligned}\sigma_0 &= [\texttt{x} \mapsto 1] \\ \sigma_1 &= [\texttt{x} \mapsto 1, \texttt{y} \mapsto 2]\end{aligned}$$

We can also write the above derivation as a single multi-step evaluation:

$$\langle \varnothing, \texttt{x := 1; } \{\texttt{y := 2; x + y}\}\rangle \to^* \langle \sigma_1, 3\rangle$$

It means that the expression $\texttt{x := 1; } \{\texttt{y := 2; x + y}\}$ evaluates to 3.

Big-Step vs. Small-Step Operational Semantics

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Are big-step and small-step operational semantics equivalent?

Big-Step vs. Small-Step Operational Semantics

Are big-step and small-step operational semantics equivalent?

No! Consider the following expression:

$$\texttt{x := 1; } \{ \texttt{ x := 2; x } \} \texttt{ + x}$$

- It evaluates to 3 under the big-step operational semantics, but
- it evaluates to 4 under the small-step operational semantics.

Big-Step vs. Small-Step Operational Semantics

Are big-step and small-step operational semantics equivalent?

No! Consider the following expression:

$$\texttt{x := 1; \{ x := 2; x \} + x}$$

- It evaluates to 3 under the big-step operational semantics, but
- it evaluates to 4 under the small-step operational semantics.

Why?

$$\frac{\dfrac{\dots}{[\texttt{x} \mapsto 1] \vdash \texttt{x := 2; x} \Rightarrow 2} \qquad \dfrac{\dots}{[\texttt{x} \mapsto 1] \vdash \texttt{x} \Rightarrow 1}}{[\texttt{x} \mapsto 1] \vdash \{\texttt{x := 2; x}\} \texttt{ + x} \Rightarrow 3}$$

vs.

$$\langle [\texttt{x} \mapsto 1], \{\texttt{x := 2; x}\} \texttt{ + x} \rangle \rightarrow \langle [\texttt{x} \mapsto 2], \texttt{x + x} \rangle \rightarrow^* \langle [\texttt{x} \mapsto 2], 4 \rangle$$

Big-Step vs. Small-Step Operational Semantics

To make big-step and small-step operational semantics equivalent, we need to use **substitution** instead of **environments** in the small-step operational semantics:

$$\frac{}{\langle x := n;\ e_2 \rangle \to \langle e_2[x \mapsto n] \rangle}$$

where $e[x \mapsto y]$ is the expression obtained by substituting x with y in e.

Big-Step vs. Small-Step Operational Semantics

To make big-step and small-step operational semantics equivalent, we need to use **substitution** instead of **environments** in the small-step operational semantics:

$$\frac{}{\langle x \ := n;\ e_2 \rangle \to \langle e_2[x \mapsto n] \rangle}$$

where $e[x \mapsto y]$ is the expression obtained by substituting x with y in e.

With substitution, the above expression evaluates to 3 under the small-step operational semantics as well:

$$\langle \mathtt{x} \ := 1;\ \{\mathtt{x} \ := 2;\ \mathtt{x}\} + \mathtt{x} \rangle \to \langle \{\mathtt{x} \ := 2;\ \mathtt{x}\} + 1 \rangle \to \langle 2 + 1 \rangle \to 3$$

Big-Step vs. Small-Step Operational Semantics

To make big-step and small-step operational semantics equivalent, we need to use **substitution** instead of **environments** in the small-step operational semantics:

$$\frac{}{\langle x \texttt{ := } n;\ e_2 \rangle \to \langle e_2[x \mapsto n] \rangle}$$

where $e[x \mapsto y]$ is the expression obtained by substituting x with y in e.

With substitution, the above expression evaluates to 3 under the small-step operational semantics as well:

$$\langle \texttt{x := 1;}\ \{\texttt{x := 2;}\ \texttt{x}\} \texttt{ + x} \rangle \to \langle \{\texttt{x := 2;}\ \texttt{x}\} \texttt{ + 1} \rangle \to \langle \texttt{2 + 1} \rangle \to 3$$

This demonstrates that representing **lexical scope** through substitution is often more intuitive than using environments when defining small-step operational semantics.

We will discuss substitution in more detail in the next lecture.

Contents

Soundness

Let's prove the **soundness** of the small-step operational semantics of VAE.

- **Soundness**: Every configuration evaluates to an integer:

$$\forall \sigma \in \Sigma.\ \forall e \in \mathbb{E}.\ \exists \sigma' \in \Sigma.\ \exists n' \in \mathbb{Z}.\ \langle \sigma, e \rangle \to^* \langle \sigma', n' \rangle$$

Is it true?

Soundness

Let's prove the **soundness** of the small-step operational semantics of VAE.

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$$\forall \sigma \in \Sigma.\ \forall e \in \mathbb{E}.\ \exists \sigma' \in \Sigma.\ \exists n' \in \mathbb{Z}.\ \langle \sigma, e \rangle \to^* \langle \sigma', n' \rangle$$

Is it true? **No!** Consider the following configuration:

$$\langle \varnothing, \mathtt{x} \rangle \not\to$$

where $\langle \sigma, e \rangle \not\to$ is shorthand for:

$$\neg(\exists \sigma' \in \Sigma.\ \exists e' \in \mathbb{E}.\ \langle \sigma, e \rangle \to \langle \sigma', e' \rangle)$$

Well-Formed Configurations

We can fix the soundness issue by restricting the evaluation to **well-formed** configurations.

Definition (Well-Formed Expressions)

A configuration $\langle \sigma, e \rangle$ is **well-formed** if and only if $\mathsf{fvs}(e) \subseteq \mathsf{dom}(\sigma)$

where $\mathsf{fvs}(e)$ is the set of **free variables** in e defined as follows:

$$
\begin{array}{lcl}
\mathsf{fvs}(n) & \triangleq & \varnothing \\
\mathsf{fvs}(e_1 \texttt{ + } e_2) & \triangleq & \mathsf{fvs}(e_1) \cup \mathsf{fvs}(e_2) \\
\mathsf{fvs}(e_1 \texttt{ * } e_2) & \triangleq & \mathsf{fvs}(e_1) \cup \mathsf{fvs}(e_2) \\
\mathsf{fvs}(x \texttt{ := } e_1\texttt{; } e_2) & \triangleq & \mathsf{fvs}(e_1) \cup (\mathsf{fvs}(e_2) \setminus \{x\}) \\
\mathsf{fvs}(x) & \triangleq & \{x\}
\end{array}
$$

Soundness

With the well-formed configurations, we can restore the soundness of the small-step operational semantics of VAE:

Theorem (Soundness)

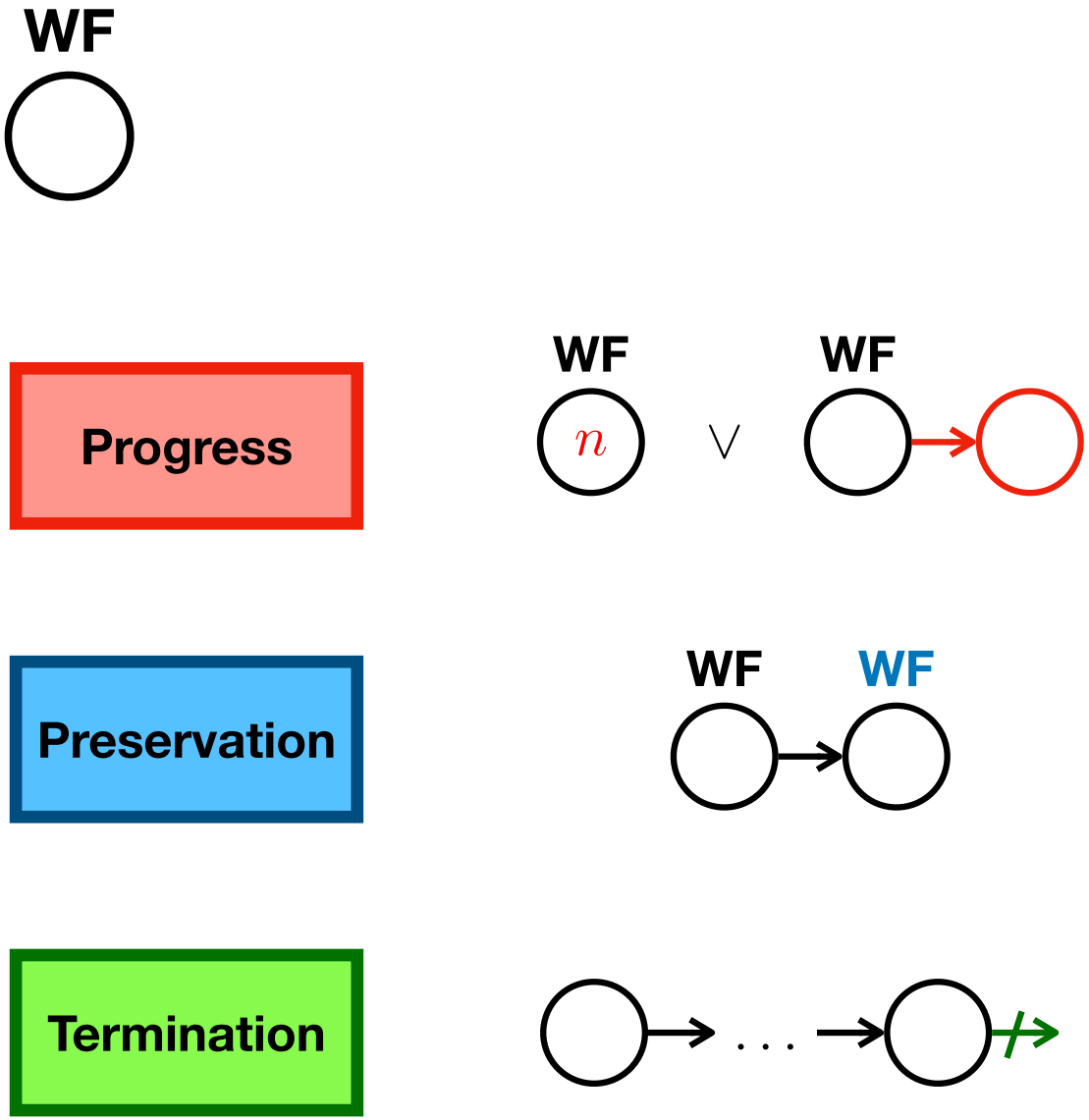
Every well-formed configuration evaluates to an integer:

$$\begin{aligned}&\forall \sigma \in \Sigma.\ \forall e \in \mathbb{E}.\\ &\quad \mathit{fvs}(e) \subseteq \mathit{dom}(\sigma) \implies\\ &\quad \exists \sigma' \in \Sigma.\ \exists n' \in \mathbb{Z}.\ \langle \sigma, e \rangle \to^* \langle \sigma', n' \rangle\end{aligned}$$

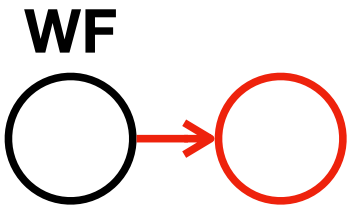
We will prove with the following three properties:

- **Progress**: Every well-formed configuration either is an integer or can take a step.
- **Preservation**: Every step preserves well-formedness of configurations.
- **Termination**: Every configuration terminates in a finite steps.

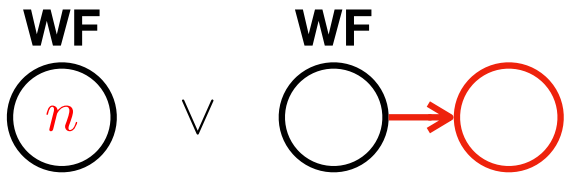
Soundness



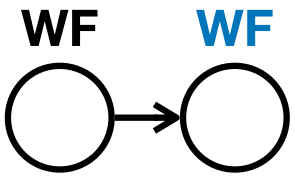
Soundness



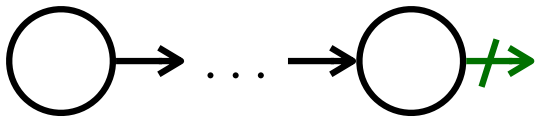
Progress



Preservation



Termination



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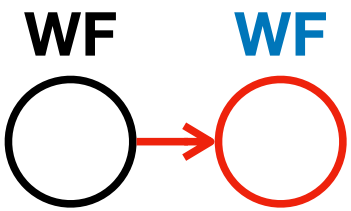

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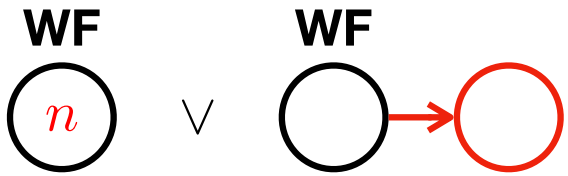

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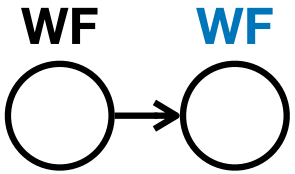

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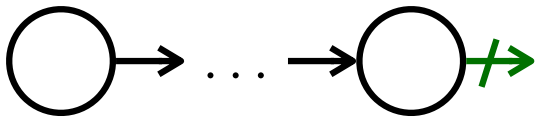

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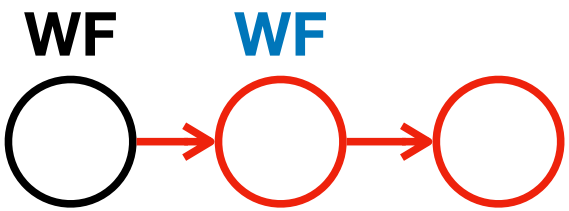

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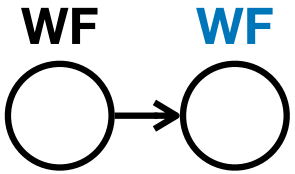

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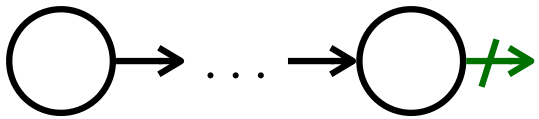

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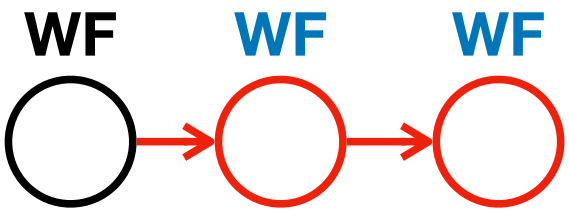

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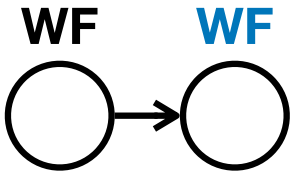

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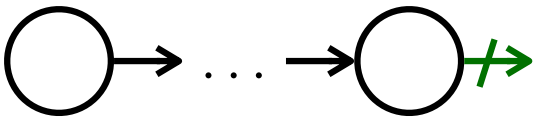

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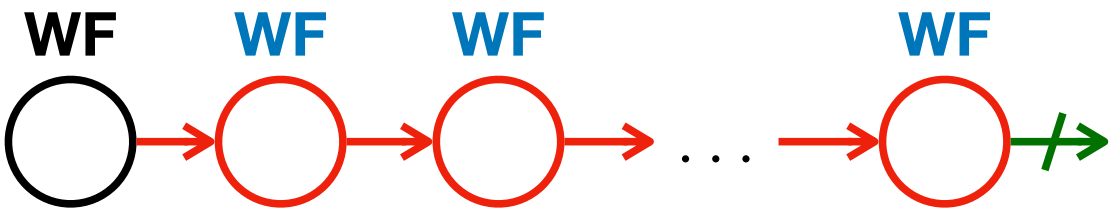

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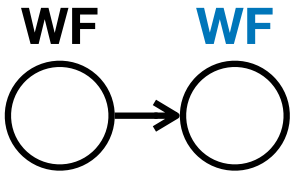

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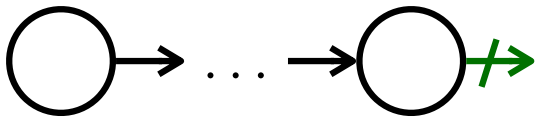

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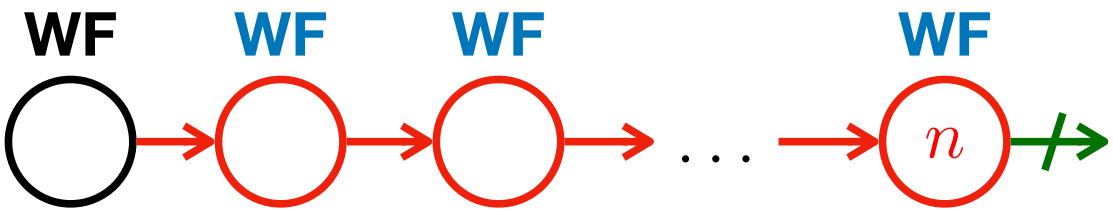

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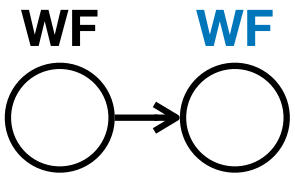

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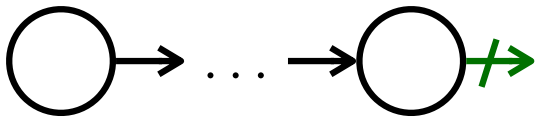

Progress

Theorem (Progress)

Every well-formed configuration either is an integer or can take a step:

$$\begin{aligned}&\forall \sigma \in \Sigma.\ \forall e \in \mathbb{E}.\\ &\quad \mathit{fvs}(e) \subseteq \mathit{dom}(\sigma) \implies\\ &\quad e \in \mathbb{Z} \vee (\exists \sigma' \in \Sigma.\ \exists e' \in \mathbb{E}.\ \langle \sigma, e \rangle \to \langle \sigma', e' \rangle)\end{aligned}$$

Let's prove it by structural induction on expressions.

- $\boxed{e = n}$: e is an integer, so the progress property holds.

Progress

PLRG

Theorem (Progress)

Every well-formed configuration either is an integer or can take a step:

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Let's prove it by structural induction on expressions.

- $\boxed{e = n}$: e is an integer, so the progress property holds.
- $\boxed{e = x}$: Since $\mathsf{fvs}(e) = \{x\} \subseteq \mathsf{dom}(\sigma)$, we have $x \in \mathsf{dom}(\sigma)$. Thus, $\langle \sigma, x \rangle \to \langle \sigma, \sigma(x) \rangle$.

$$\frac{x \in \mathsf{dom}(\sigma)}{\langle \sigma, x \rangle \to \langle \sigma, \sigma(x) \rangle}$$

Progress

- $\boxed{e = e_1 + e_2}$: We know that $\langle \sigma, e_1 \rangle$ and $\langle \sigma, e_2 \rangle$ are well-formed:

$$\mathsf{fvs}(e_1) \subseteq \mathsf{fvs}(e_1 + e_2) \subseteq \mathsf{dom}(\sigma)$$
$$\mathsf{fvs}(e_2) \subseteq \mathsf{fvs}(e_1 + e_2) \subseteq \mathsf{dom}(\sigma)$$

because $\mathsf{fvs}(e_1 + e_2) \triangleq \mathsf{fvs}(e_1) \cup \mathsf{fvs}(e_2)$.

Progress

- $\boxed{e = e_1 \text{ + } e_2}$: We know that $\langle \sigma, e_1 \rangle$ and $\langle \sigma, e_2 \rangle$ are well-formed:

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  because $\mathsf{fvs}(e_1 \text{ + } e_2) \triangleq \mathsf{fvs}(e_1) \cup \mathsf{fvs}(e_2)$.

  - $\boxed{e_1 \notin \mathbb{Z}}$: By the I.H., we have $\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle$ for some $\sigma' \in \Sigma$ and $e_1' \in \mathbb{E}$. Thus, $\langle \sigma, e_1 \text{ + } e_2 \rangle \to \langle \sigma', e_1' \text{ + } e_2 \rangle$.

$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 \text{ + } e_2 \rangle \to \langle \sigma', e_1' \text{ + } e_2 \rangle}$$

Progress

- $\boxed{e = e_1 \text{ + } e_2}$: We know that $\langle \sigma, e_1 \rangle$ and $\langle \sigma, e_2 \rangle$ are well-formed:

$$\mathsf{fvs}(e_1) \subseteq \mathsf{fvs}(e_1 \text{ + } e_2) \subseteq \mathsf{dom}(\sigma)$$
$$\mathsf{fvs}(e_2) \subseteq \mathsf{fvs}(e_1 \text{ + } e_2) \subseteq \mathsf{dom}(\sigma)$$

  because $\mathsf{fvs}(e_1 \text{ + } e_2) \triangleq \mathsf{fvs}(e_1) \cup \mathsf{fvs}(e_2)$.

  - $\boxed{e_1 \notin \mathbb{Z}}$: By the I.H., we have $\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle$ for some $\sigma' \in \Sigma$ and $e_1' \in \mathbb{E}$. Thus, $\langle \sigma, e_1 \text{ + } e_2 \rangle \to \langle \sigma', e_1' \text{ + } e_2 \rangle$.

$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 \text{ + } e_2 \rangle \to \langle \sigma', e_1' \text{ + } e_2 \rangle}$$

  - $\boxed{e_1 \in \mathbb{Z} \wedge e_2 \notin \mathbb{Z}}$: Similarly, we can use I.H. for $\langle \sigma, e_2 \rangle$.

$$\frac{\langle \sigma, e_2 \rangle \to \langle \sigma', e_2' \rangle}{\langle \sigma, n_1 \text{ + } e_2 \rangle \to \langle \sigma', n_1 \text{ + } e_2' \rangle}$$

Progress

- $\boxed{e = e_1 + e_2}$: We know that $\langle \sigma, e_1 \rangle$ and $\langle \sigma, e_2 \rangle$ are well-formed:

$$\mathsf{fvs}(e_1) \subseteq \mathsf{fvs}(e_1 + e_2) \subseteq \mathsf{dom}(\sigma)$$
$$\mathsf{fvs}(e_2) \subseteq \mathsf{fvs}(e_1 + e_2) \subseteq \mathsf{dom}(\sigma)$$

  because $\mathsf{fvs}(e_1 + e_2) \triangleq \mathsf{fvs}(e_1) \cup \mathsf{fvs}(e_2)$.

  - $\boxed{e_1 \notin \mathbb{Z}}$: By the I.H., we have $\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle$ for some $\sigma' \in \Sigma$ and $e_1' \in \mathbb{E}$. Thus, $\langle \sigma, e_1 + e_2 \rangle \to \langle \sigma', e_1' + e_2 \rangle$.

$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \to \langle \sigma', e_1' + e_2 \rangle}$$

  - $\boxed{e_1 \in \mathbb{Z} \wedge e_2 \notin \mathbb{Z}}$: Similarly, we can use I.H. for $\langle \sigma, e_2 \rangle$.

$$\frac{\langle \sigma, e_2 \rangle \to \langle \sigma', e_2' \rangle}{\langle \sigma, n_1 + e_2 \rangle \to \langle \sigma', n_1 + e_2' \rangle}$$

  - $\boxed{e_1 \in \mathbb{Z} \wedge e_2 \in \mathbb{Z}}$: Since there is no premise, it can always take a step.

$$\frac{}{\langle \sigma, n_1 + n_2 \rangle \to \langle \sigma, n_1 + n_2 \rangle}$$

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  - $\boxed{e_1 \in \mathbb{Z}}$: Since there is no premise, it can always take a step.

$$\frac{}{\langle \sigma, x \ := n;\ e_2 \rangle \to \langle \sigma[x \mapsto n], e_2 \rangle}$$

Preservation

Theorem (Preservation)

Every step preserves well-formedness of configurations:

$$\forall \sigma, \sigma' \in \Sigma.\ \forall e, e' \in \mathbb{E}.$$
$$\mathit{fvs}(e) \subseteq \mathit{dom}(\sigma) \land \langle \sigma, e \rangle \to \langle \sigma', e' \rangle \implies \mathit{fvs}(e') \subseteq \mathit{dom}(\sigma')$$

Preservation

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Let's prove it by induction on the derivation of inference rules.

- $\boxed{e = (x := n;\ e_2)}$: See the following inference rule:

$$\frac{}{\langle \sigma, x := n;\ e_2 \rangle \to \langle \sigma[x \mapsto n], e_2 \rangle}$$

We can show that $\mathsf{fvs}(e_2) \subseteq \mathsf{dom}(\sigma[x \mapsto n])$ as follows:

$$\begin{aligned} \mathsf{fvs}(e_2) &\subseteq \mathsf{fvs}(e_2) \setminus \{x\} \cup \{x\} \\ &= \mathsf{fvs}(x := n;\ e_2) \cup \{x\} \\ &\subseteq \mathsf{dom}(\sigma) \cup \{x\} \\ &= \mathsf{dom}(\sigma[x \mapsto n]) \end{aligned}$$

Preservation

- $\boxed{e = e_1 + e_2}$: We know that $\langle \sigma, e_1 \rangle$ and $\langle \sigma, e_2 \rangle$ are well-formed:

$$\mathsf{fvs}(e_1) \subseteq \mathsf{fvs}(e_1 + e_2) \subseteq \mathsf{dom}(\sigma)$$
$$\mathsf{fvs}(e_2) \subseteq \mathsf{fvs}(e_1 + e_2) \subseteq \mathsf{dom}(\sigma)$$

because $\mathsf{fvs}(e_1 + e_2) \triangleq \mathsf{fvs}(e_1) \cup \mathsf{fvs}(e_2)$.

$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \to \langle \sigma', e_1' + e_2 \rangle}$$

Thus, I.H. tells us that $\mathsf{fvs}(e_1') \subseteq \mathsf{dom}(\sigma')$.

We know $\mathsf{dom}(\sigma) \subseteq \mathsf{dom}(\sigma')$ because all small-step evaluation rules only add new bindings to environments.

Thus, $\langle \sigma', e_1' + e_2 \rangle$ is well-formed because:

$$\mathsf{fvs}(e_1' + e_2) = \mathsf{fvs}(e_1') \cup \mathsf{fvs}(e_2) \subseteq \mathsf{dom}(\sigma') \cup \mathsf{dom}(\sigma) \subseteq \mathsf{dom}(\sigma')$$

- We can prove the rest of the cases similarly.

Termination

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Theorem (Termination)

Every configuration terminates in a finite steps:

$$\forall \sigma, \sigma' \in \Sigma.\ \forall e, e' \in \mathbb{E}.\ \exists k \in \mathbb{N}.\ \langle \sigma, e \rangle \rightarrow^{k} \langle \sigma', e' \rangle \not\rightarrow$$

Termination

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We can prove it by showing that each small-step evaluation strictly decreases the **size** of expressions:

$$\begin{array}{lcl}
\mathsf{size}(n) & \triangleq & 0 \\
\mathsf{size}(e_1 \texttt{ + } e_2) & \triangleq & 1 + \mathsf{size}(e_1) + \mathsf{size}(e_2) \\
\mathsf{size}(e_1 \texttt{ * } e_2) & \triangleq & 1 + \mathsf{size}(e_1) + \mathsf{size}(e_2) \\
\mathsf{size}(x \texttt{ := } e_1\texttt{; } e_2) & \triangleq & 1 + \mathsf{size}(e_1) + \mathsf{size}(e_2) \\
\mathsf{size}(x) & \triangleq & 1
\end{array}$$

Based on the induction on the derivation of inference rules, we can show that $\mathsf{size}(e) > \mathsf{size}(e')$ whenever $\langle \sigma, e \rangle \to \langle \sigma', e' \rangle$.

Soundness vs. Completeness

- **Soundness**: Every well-formed configuration evaluates to an integer.
- **Completeness**: Every configuration that evaluates to an integer is well-formed.

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We have proved the soundness of the small-step operational semantics of VAE, but we have not proved its completeness.

Is it complete?

Soundness vs. Completeness

- **Soundness**: Every well-formed configuration evaluates to an integer.
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We have proved the soundness of the small-step operational semantics of VAE, but we have not proved its completeness.

Is it complete? **No!** Consider the following configuration evaluating to 2:

$$\langle \varnothing, \{\mathtt{x} := 1;\ \mathtt{x}\} + \mathtt{x} \rangle \to \langle [\mathtt{x} \mapsto 1], \mathtt{x} + \mathtt{x} \rangle \to^* \langle [\mathtt{x} \mapsto 1], 2 \rangle$$

But, this configuration is not well-formed:

$$\mathsf{fvs}((\mathtt{x} := 1;\ \mathtt{x}) + \mathtt{x}) = \{\mathtt{x}\} \not\subseteq \mathsf{dom}(\varnothing) = \varnothing$$

It happens because $\mathsf{fvs}(-)$ follows lexical scope, while the small-step operational semantics of VAE with environments does not.

Contents

Simple Imperative Language – IMP

PLRG

$$
\begin{array}{lll}
\text{Expressions} & e & ::= n \mid x \mid e \texttt{ + } e \mid e \texttt{ * } e \mid e \texttt{ < } e \mid \texttt{true} \mid \texttt{false} \\
\text{Statements} & s & ::= \texttt{skip} \\
 & & \mid x \texttt{ := } e \\
 & & \mid s \texttt{;} s \\
 & & \mid \texttt{if } e \texttt{ then } s \texttt{ else } s \\
 & & \mid \texttt{while } e \texttt{ do } s \\
\text{Values} & v & ::= n \mid \texttt{true} \mid \texttt{false}
\end{array}
$$

where $n \in \mathbb{Z}$, and $x \in \mathbb{X}$.

We will define two forms of small-step operational semantics for IMP:

$$\boxed{\langle \sigma, e \rangle \to \langle \sigma, v \rangle} \qquad \boxed{\langle \sigma, s \rangle \to \langle \sigma, s \rangle}$$

where $\sigma : \mathbb{X} \rightharpoonup \mathbb{V}$.

Expressions

$$\frac{x \in \mathsf{dom}(\sigma)}{\langle \sigma, x \rangle \to \langle \sigma, \sigma(x) \rangle}$$

$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma, e_1' \rangle}{\langle \sigma, e_1 \texttt{ + } e_2 \rangle \to \langle \sigma, e_1' \texttt{ + } e_2 \rangle} \qquad \frac{\langle \sigma, e_2 \rangle \to \langle \sigma, e_2' \rangle}{\langle \sigma, n_1 \texttt{ + } e_2 \rangle \to \langle \sigma, n_1 \texttt{ + } e_2' \rangle} \qquad \frac{}{\langle \sigma, n_1 \texttt{ + } n_2 \rangle \to \langle \sigma, n_1 + n_2 \rangle}$$

$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma, e_1' \rangle}{\langle \sigma, e_1 \texttt{ * } e_2 \rangle \to \langle \sigma, e_1' \texttt{ * } e_2 \rangle} \qquad \frac{\langle \sigma, e_2 \rangle \to \langle \sigma, e_2' \rangle}{\langle \sigma, n_1 \texttt{ * } e_2 \rangle \to \langle \sigma, n_1 \texttt{ * } e_2' \rangle} \qquad \frac{}{\langle \sigma, n_1 \texttt{ * } n_2 \rangle \to \langle \sigma, n_1 \times n_2 \rangle}$$

$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma, e_1' \rangle}{\langle \sigma, e_1 \texttt{ < } e_2 \rangle \to \langle \sigma, e_1' \texttt{ < } e_2 \rangle} \qquad \frac{\langle \sigma, e_2 \rangle \to \langle \sigma, e_2' \rangle}{\langle \sigma, n_1 \texttt{ < } e_2 \rangle \to \langle \sigma, n_1 \texttt{ < } e_2' \rangle} \qquad \frac{}{\langle \sigma, n_1 \texttt{ < } n_2 \rangle \to \langle \sigma, n_1 < n_2 \rangle}$$

Statements

$$\frac{\langle \sigma, e \rangle \to \langle \sigma, e' \rangle}{\langle \sigma, x \texttt{ := } e \rangle \to \langle \sigma, x \texttt{ := } e' \rangle} \qquad \frac{}{\langle \sigma, x \texttt{ := } v \rangle \to \langle \sigma[x \mapsto v], \texttt{skip} \rangle}$$

$$\frac{\langle \sigma, s_1 \rangle \to \langle \sigma, s_1' \rangle}{\langle \sigma, s_1 \texttt{ ; } s_2 \rangle \to \langle \sigma, s_1' \texttt{ ; } s_2 \rangle} \qquad \frac{}{\langle \sigma, \texttt{skip; } s_2 \rangle \to \langle \sigma, s_2 \rangle}$$

$$\frac{\langle \sigma, e \rangle \to \langle \sigma, e' \rangle}{\langle \sigma, \texttt{if } e \texttt{ then } s_1 \texttt{ else } s_2 \rangle \to \langle \sigma, \texttt{if } e' \texttt{ then } s_1 \texttt{ else } s_2 \rangle}$$

$$\frac{}{\langle \sigma, \texttt{if true then } s_1 \texttt{ else } s_2 \rangle \to \langle \sigma, s_1 \rangle}$$

$$\frac{}{\langle \sigma, \texttt{if false then } s_1 \texttt{ else } s_2 \rangle \to \langle \sigma, s_2 \rangle}$$

$$\frac{}{\langle \sigma, \texttt{while } e \texttt{ do } s \rangle \to \langle \sigma, \texttt{if } e \texttt{ then } (s \texttt{;while } e \texttt{ do } s) \texttt{ else skip} \rangle}$$

Next Lecture

- Evaluation Context and Lambda Calculus

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