Lecture 6 – Fixed-Point Theory

AAA551: Programming Language Theory

Jihyeok Park

PLRG

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Recall

- Denotational Semantics
- Simple Imperative Language – IMP
    - Expressions
    - Statements
    - Solving Recursive Equations
    - `while` Statement Revisited
    - Explicit Errors
- Non-Deterministic Imperative Language – NIMP
    - Expressions
    - Statements

`while` Statement

$$S[\![\texttt{while}\ e\ \texttt{do}\ s]\!] \triangleq \mathbf{lfp}(F) = \bigcup_{n \in \mathbb{Z}} F^n(\varnothing)$$

where

$$F(f)(\sigma) = \begin{cases} \sigma & \text{if } E[\![e]\!](\sigma) = \texttt{false} \\ f(S[\![s]\!](\sigma)) & \text{if } E[\![e]\!](\sigma) = \texttt{true} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Contents

1. Partial Orders and Lattices
    Partial Orders
    Complete Partial Orders (CPOs)
    Lattices
    Posets vs. CPOs vs. Lattices

2. Fixpoint Theory
    Monotone and Continuous Functions
    Fixpoints
    Tarski's Fixpoint Theorem
    Kleene's Fixpoint Theorem
    `while` Statements in IMP
    `while` Statements in NIMP

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    - Tarski's Fixpoint Theorem
    - Kleene's Fixpoint Theorem
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    - `while` Statements in NIMP

Partial Orders

Definition (Partial Orders)

A *partial order* is a binary relation $\sqsubseteq$ on a set S that is:

- **reflexive**: $\forall x \in S, x \sqsubseteq x$
- **antisymmetric**: $\forall x, y \in S, (x \sqsubseteq y \land y \sqsubseteq x) \implies x = y$
- **transitive**: $\forall x, y, z \in S, (x \sqsubseteq y \land y \sqsubseteq z) \implies x \sqsubseteq z$.

We call the pair $(S, \sqsubseteq)$ a *partially ordered set* (or *poset*).

We use the notation $x \sqsubset y \triangleq (x \sqsubseteq y \land x \neq y)$.

Example

$(\mathcal{P}(X), \subseteq)$ is a poset for any set X.

- **reflexive**: $\forall A \subseteq \mathbb{Z}, A \subseteq A$
- **antisymmetric**: $\forall A, B \subseteq \mathbb{Z}, (A \subseteq B \land B \subseteq A) \implies A = B$
- **transitive**: $\forall A, B, C \subseteq \mathbb{Z}, (A \subseteq B \land B \subseteq C) \implies A \subseteq C$

Partial Orders – Hasse Diagrams

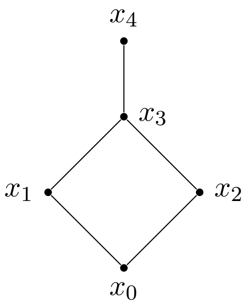
We often use *Hasse diagrams* to visually represent posets.

A poset $(S, \sqsubseteq)$ is defined as follows:

- $S \triangleq \{x_0, x_1, x_2, x_3, x_4\}$
- $\sqsubseteq$ is the smallest partial order on S such that:

$$x_0 \sqsubseteq x_1$$
$$x_0 \sqsubseteq x_2$$
$$x_1 \sqsubseteq x_3$$
$$x_2 \sqsubseteq x_3$$
$$x_3 \sqsubseteq x_4$$

Hasse diagram of $(S, \sqsubseteq)$:



By definition of reflexivity, $x_i \sqsubseteq x_i$ for all $0 \leq i \leq 4$.

By definition of transitivity,

- $x_0 \sqsubseteq x_3 \qquad (\because x_0 \sqsubseteq x_1 \sqsubseteq x_3)$
- $x_2 \sqsubseteq x_4 \qquad (\because x_2 \sqsubseteq x_3 \sqsubseteq x_4)$

Total Orders

Definition (Total Orders)

A *total order* is a partial order $\sqsubseteq$ on a set S such that:

- **total**: $\forall x, y \in S, (x \sqsubseteq y \lor y \sqsubseteq x)$

Example

$(\mathbb{Z}, \leq)$ is a total order.

- **reflexive**: $\forall n \in \mathbb{Z}, n \leq n$
- **antisymmetric**: $\forall m, n \in \mathbb{Z}, (m \leq n \land n \leq m) \implies m = n$
- **transitive**: $\forall m, n, p \in \mathbb{Z}, (m \leq n \land n \leq p) \implies m \leq p$
- **total**: $\forall m, n \in \mathbb{Z}, (m \leq n \lor n \leq m)$

Example

$(\mathcal{P}(S), \sqsubseteq)$ is a poset but not total.

- $\{x\}$ and $\{y\}$ are incomparable if $x \neq y$ and $x, y \in S$.

Minimum and Maximum Elements

Definition (Minimum and Maximum Elements)

Let $(S, \sqsubseteq)$ be a poset and $x \in S$. Then, x is:

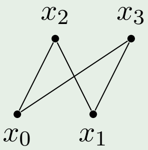
- the *maximum element* of S if $\forall y \in S, y \sqsubseteq x$
- the *minimum element* of S if $\forall y \in S, x \sqsubseteq y$

We often use the following notations:

- $\top$ (**"top"**) to denote the maximum element
- $\bot$ (**"bottom"**) to denote the minimum element

Example

The following poset does not have the minimum or maximum element.



Bounds

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Definition (Bounds)

Let $(S, \sqsubseteq)$ be a poset and $T \subseteq S$. Then, $x \in S$ is:

- an *upper bound* of T if:

$$\forall t \in T, t \sqsubseteq x$$

- the *least upper bound* (lub or **join**) of T if:

$$\forall t \in T, t \sqsubseteq x \land \forall y \in S, (\forall t \in T, t \sqsubseteq y) \implies x \sqsubseteq y$$

Dually, we define *lower bounds* and *greatest lower bounds* (glb or **meet**).

Not all subsets have lubs or glbs.

If T has the lub (or glb), it is unique and we denote it as $\bigsqcup T$ (or $\bigsqcap T$).

We use the notations $x \sqcup y \triangleq \bigsqcup\{x, y\}$ and $x \sqcap y \triangleq \bigsqcap\{x, y\}$.

Chains

Definition (Chains)

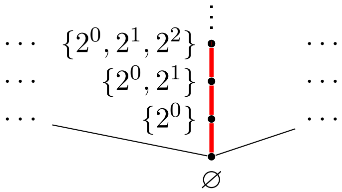
Let $(S, \sqsubseteq)$ be a poset. A subset $C \subseteq S$ is a *chain* if $(C, \sqsubseteq)$ is a total, i.e.,

$$\forall x, y \in C, (x \sqsubseteq y \vee y \sqsubseteq x)$$

Example

In the poset $(\mathcal{P}(\mathbb{N}), \subseteq)$, the following subset $C \subseteq \mathcal{P}(\mathbb{N})$ is an infinite chain:

$$C = \{c_i \mid i \in \mathbb{N}\} \text{ where } \forall i \in \mathbb{N}, c_i = \{2^j \mid 0 \leq j < i\}$$



Complete Partial Orders

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Definition (Complete Partial Orders)

A *complete partial order* (CPO) is a poset $(S, \sqsubseteq)$ if every chain $C \subseteq S$ has the lub $\bigsqcup C$.

Lemma

Let $(S, \sqsubseteq)$ *be a CPO. Then,* S *has the minimum element* $\bot$.

The empty set $\varnothing$ is a chain because it is vacuously total.

Since the lub of $\varnothing$ is the minimum element $\bot$ (i.e., $\bigsqcup \varnothing = \bot$), S has the minimum element $\bot$.

Lattices

Definition (Lattices)

A *lattice* $(S, \sqsubseteq, \sqcup, \sqcap)$ is a poset such that:

$$\forall x, y \in S, x \sqcup y \text{ and } x \sqcap y \text{ exist in } S$$

All finite subsets $T \subseteq S$ have lubs and glbs because we can compute them by iteratively applying $\sqcup$ and $\sqcap$ to the elements of T.

Definition (Complete Lattices)

A *complete lattice* $(S, \sqsubseteq, \sqcup, \sqcap, \top, \bot)$ is a poset such that:

$$\forall T \subseteq S, \bigsqcup T \text{ and } \bigsqcap T \text{ exist in } S$$

Every complete lattice:

- has the **maximum** $\top = \bigsqcup S = \bigsqcap \varnothing$ and **minimum** $\bot = \bigsqcap S = \bigsqcup \varnothing$.
- is **lattice** because every pair is a subset.
- is a **CPO** because every chain is a subset.

Posets vs. CPOs and Posets vs. Lattices

Not all posets are CPOs or lattices.

Example

$(\mathbb{N}, \leq_2)$ is a poset where $\leq_2$ is defined as follows:

$$\forall x, y \in \mathbb{N}. \quad x \leq_2 y \quad \triangleq \quad (x \leq y) \land (x \equiv y \pmod 2)$$

but it is neither a CPO nor a lattice.

It is not a CPO:

- $C = \{0, 2, 4, \dots\}$ is a chain because it is a total order.
- C does not have the lub in $\mathbb{N}$.

It is not a lattice:

- 0 and 1 do not have the lub in $\mathbb{N}$.
- 0 and 1 do not have the glb in $\mathbb{N}$.

CPOs vs. Lattices

Not all CPOs are lattices.

Example

$(X \rightharpoonup Y, \sqsubseteq)$ for *partial functions* from X to Y is a CPO where $f \sqsubseteq g$ if:

$$\mathsf{dom}(f) \subseteq \mathsf{dom}(g) \quad \land \quad \forall x \in \mathsf{dom}(f), f(x) = g(x)$$

for all $f, g \in X \rightharpoonup Y$, but it is not a lattice.

The minimum element $\bot$ is the empty function $\varnothing$.

It is a CPO because we can define lub of any chain C as follows:

$$(\bigsqcup C)(x) = \begin{cases} f(x) & \text{if } \exists f \in C.\ x \in \mathsf{dom}(f) \\ \text{undefined} & \text{otherwise} \end{cases}$$

It is not a lattice because f and g do not have lub in $X \rightharpoonup Y$:

$$f = \{(x, y)\} \quad \text{and} \quad g = \{(x, y')\}$$

where $x \in X$ and $y, y' \in Y$ such that $y \neq y'$.

Lattices vs. CPOs

Not all lattices are CPOs.

Example

$(\mathbb{Z}^+, |)$ is a lattice where $|$ is the divisibility relation defined as:

$$\forall m, n \in \mathbb{Z}^+. \quad m \mid n \quad \triangleq \quad \exists k \in \mathbb{Z}^+.\ n = m \cdot k$$

but it is not a CPO.

The minimum element $\bot$ is 1 because $1 \mid n$ for all $n \in \mathbb{Z}^+$.

It is a lattice because we can define:

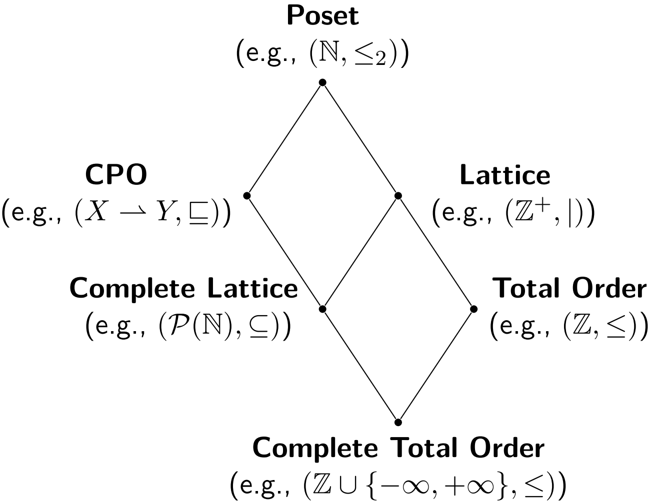
- $m \sqcup n = \mathrm{lcm}(m, n)$ (least common multiple)
- $m \sqcap n = \gcd(m, n)$ (greatest common divisor)

It is not a CPO because the following chain C does not have the lub in $\mathbb{Z}^+$:

$$C = \{2^i \mid i \in \mathbb{N}\}$$

Summary

The following Hasse diagram summarizes the implications between different classes of posets:



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Monotone Functions

Definition (Monotone Functions)

Let $(S, \sqsubseteq)$ be a poset. A function $f : S \to S$ is *monotone* if:

$$\forall x, y \in S, x \sqsubseteq y \implies f(x) \sqsubseteq f(y)$$

$$\begin{array}{ccc} x & \xrightarrow{\quad f \quad} & f(x) \\ \rotatebox{-90}{$\sqsubseteq$} & & \rotatebox{-90}{$\sqsubseteq$} \\ y & \xrightarrow[\quad f \quad]{} & f(y) \end{array}$$

Example

Let $(\mathcal{P}(\mathbb{N}), \subseteq)$ be a poset. Then, the following functions are monotone:

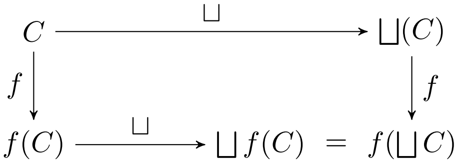
- $f(A) = A \cup \{42\}$
- $f(A) = A \cap S$ for some fixed $S \subseteq \mathbb{N}$

Continuous Functions

Definition (Continuous Functions)

Let $(S, \sqsubseteq)$ be a poset. A function $f : S \to S$ is (Scott) *continuous* if:

$$\forall C \subseteq S. \quad C \text{ is a non-empty chain} \implies f(\bigsqcup C) = \bigsqcup f(C)$$



Example

Previously defined monotone functions on $(\mathcal{P}(\mathbb{N}), \subseteq)$ are also continuous, but the following function is not continuous:

$$f(A) = \begin{cases} \varnothing & \text{if } A \text{ is finite} \\ A & \text{otherwise} \end{cases}$$

Monotone vs. Continuous Functions

Lemma

Function composition preserves monotonicity and continuity:

$$\forall f, g : S \to S. \quad \begin{cases} f \text{ and } g \text{ are monotone} \implies f \circ g \text{ is monotone} \\ f \text{ and } g \text{ are continuous} \implies f \circ g \text{ is continuous} \end{cases}$$

For all $x, y \in S$, $x \sqsubseteq y \implies g(x) \sqsubseteq g(y) \implies f(g(x)) \sqsubseteq f(g(y))$.

Similarly, for all non-empty chains $C \subseteq S$,

$$(f \circ g)(\bigsqcup C) = f(g(\bigsqcup C)) = f(\bigsqcup g(C)) = \bigsqcup f(g(C)) = \bigsqcup (f \circ g)(C)$$

Lemma

Every continuous function $f : S \to S$ on a poset $(S, \sqsubseteq)$ is monotone.

Every pair $x, y \in S$ such that $x \sqsubseteq y$ forms a chain $\{x, y\}$.

Thus, $f(x) \sqsubseteq (f(x) \sqcup f(y)) = \bigsqcup f(\{x, y\}) = f(\bigsqcup \{x, y\}) = f(y)$.

Monotone vs. Continuous Functions

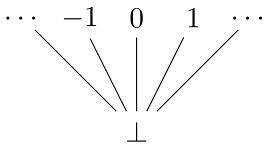
Lemma

In a monotone function $f : S \to S$ on a poset $(S, \sqsubseteq)$, if all chains are finite, then it is also continuous.

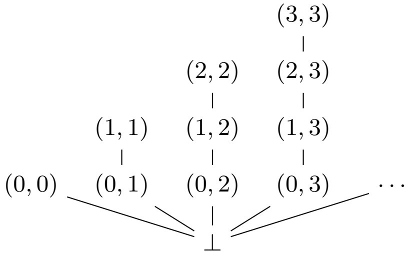
Let $C = c_0 \sqsubseteq c_1 \sqsubseteq \ldots \sqsubseteq c_n$ be a non-empty finite chain in S.

Then, $f(c_0) \sqsubseteq f(c_1) \sqsubseteq \ldots \sqsubseteq f(c_n)$ because f is monotone.

Thus, $f(\bigsqcup C) = f(c_n) = \bigsqcup f(C)$.



finite height



infinite height but all chains are finite

Fixpoints

Definition (Fixpoints)

Let $(S, \sqsubseteq)$ be a poset and $f : S \to S$ be a function. Then,

- a **fixpoint** of f is an element $x \in S$ such that $f(x) = x$
- a **pre-fixpoint** of f is an element $x \in S$ such that $f(x) \sqsubseteq x$
- a **post-fixpoint** of f is an element $x \in S$ such that $x \sqsubseteq f(x)$
- the **least fixpoint** of f is the minimum among fixpoints
- the **greatest fixpoint** of f is the maximum among fixpoints

We use the notations $\mathbf{lfp}(f)$ and $\mathbf{gfp}(f)$ to denote the least and greatest fixpoints of f, respectively, if they exist.

We use the notation $\mathbf{fix}(f)$ to denote the set of all fixpoints of f:

$$\mathbf{fix}(f) = \{x \in S \mid f(x) = x\}$$

Tarski's Fixpoint Theorem

Theorem (Tarski's Fixpoint Theorem)

Let $(S, \sqsubseteq)$ be a complete lattice and $f : S \to S$ be a monotone function. Then, the following hold:

- $\mathbf{lfp}(f) = \bigsqcap\{x \in S \mid f(x) \sqsubseteq x\}$ *is the least fixpoint of* f
- $\mathbf{gfp}(f) = \bigsqcup\{x \in S \mid x \sqsubseteq f(x)\}$ *is the greatest fixpoint of* f
- $\mathbf{fix}(f)$ *is a complete lattice*

However, it has two limitations:

- It is **not constructive**. It does not provide how to compute the least or greatest fixpoint.
- We **cannot apply** it to posets that are **not complete lattices**. For example, $(X \rightharpoonup Y, \sqsubseteq)$ is a CPO but not a complete lattice.

Kleene's Fixpoint Theorem

Theorem (Kleene's Fixpoint Theorem)

Let $(S, \sqsubseteq)$ be a CPO and $f : S \to S$ be a continuous function. Then,

$$\mathbf{lfp}(f) = \bigsqcup_{n \in \mathbb{N}} f^n(\bot)$$

- $\bigsqcup_{n \in \mathbb{N}} f^n(\bot)$ **exists in** S.

  Since f is continuous, it is monotone.

  Thus, $\{f^n(\bot) \mid n \in \mathbb{N}\}$ is a chain because:

  $$\bot \sqsubseteq f(\bot) \implies f(\bot) \sqsubseteq f^2(\bot) \implies f^2(\bot) \sqsubseteq f^3(\bot) \implies \dots$$

  Since $(S, \sqsubseteq)$ is a CPO, $\bigsqcup_{n \in \mathbb{N}} f^n(\bot)$ exists in S.

Kleene's Fixpoint Theorem

- $\bigsqcup_{n\in\mathbb{N}} f^n(\bot)$ **is a fixpoint**.

$$
\begin{array}{rcll}
f(\bigsqcup_{n\in\mathbb{N}} f^n(\bot)) & = & \bigsqcup_{n\in\mathbb{N}} f(f^n(\bot)) & (\because f \text{ is continuous}) \\
& = & \bigsqcup_{n\in\mathbb{N}} f^{n+1}(\bot) & (\because f^{n+1} = f \circ f^n) \\
& = & \bot \sqcup \bigsqcup_{n\in\mathbb{N}} f^{n+1}(\bot) & (\because \bot \text{ is the minimum}) \\
& = & \bigsqcup_{n\in\mathbb{N}} f^n(\bot) &
\end{array}
$$

- $\bigsqcup_{n\in\mathbb{N}} f^n(\bot)$ **is the least fixpoint**.

Let $x \in S$ be a fixpoint of f. Then, $\bot \sqsubseteq x$ because $\bot$ is the minimum.

Thus, $f(\bot) \sqsubseteq f(x) = x$ because f is monotone.

By induction on n, we can show that $f^n(\bot) \sqsubseteq x$ for all $n \in \mathbb{N}$.

Thus, x is an upper bound of $\{f^n(\bot) \mid n \in \mathbb{N}\}$, and hence $\bigsqcup_{n\in\mathbb{N}} f^n(\bot) \sqsubseteq x$.

`while` Statements in IMP

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$$S[\![\texttt{while } e \texttt{ do } s]\!] \triangleq \textbf{lfp}(F) = \bigcup_{n \in \mathbb{N}} F^n(\varnothing)$$

where

$$F(f)(\sigma) = \begin{cases} \sigma & \text{if } E[\![e]\!](\sigma) = \texttt{false} \\ f(S[\![s]\!](\sigma)) & \text{if } E[\![e]\!](\sigma) = \texttt{true} \\ \text{undefined} & \text{otherwise} \end{cases}$$

The poset $(\Sigma \rightharpoonup \Sigma, \sqsubseteq)$ is a CPO. We need to show that $F : (\Sigma \rightharpoonup \Sigma) \to (\Sigma \rightharpoonup \Sigma)$ is continuous to apply Kleene's fixpoint theorem.

Since function composition preserves continuity, it suffices to show that the following functions are continuous:

- $E[\![e]\!] : \Sigma \rightharpoonup \mathbb{V}$ and $S[\![s]\!] : \Sigma \rightharpoonup \Sigma$
- $\lambda f.\lambda\sigma. \begin{cases} \sigma & \text{if } \neg e(\sigma) \\ f(s(\sigma)) & \text{if } e(\sigma) \\ \text{undefined} & \text{otherwise} \end{cases}$ for some $e : \Sigma \rightharpoonup \mathbb{V}$ and $s : \Sigma \rightharpoonup \Sigma$

`while` Statements in NIMP

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$$S[\![\texttt{while } e \texttt{ do } s]\!] \triangleq \mathbf{lfp}(F) = \bigcup_{n \in \mathbb{N}} F^n(\varnothing)$$

where

$$\begin{aligned} F(g)(X) \quad \triangleq \quad & g(S[\![s]\!](\{\sigma \in X \mid \texttt{true} \in E[\![e]\!](\sigma)\})) \cup \\ & \{\sigma \in X \mid \texttt{false} \in E[\![e]\!](\sigma)\} \end{aligned}$$

To apply Kleene's fixpoint theorem, we need to show that $(\Sigma \to \mathcal{P}(\Sigma), \dot{\subseteq})$ is a CPO and $F : (\Sigma \to \mathcal{P}(\Sigma)) \to (\Sigma \to \mathcal{P}(\Sigma))$ is continuous.

Similar to IMP, it suffices to show that the following functions are continuous:

- $E[\![e]\!] : \Sigma \to \mathcal{P}(\mathbb{V})$ and $S[\![s]\!] : \Sigma \to \mathcal{P}(\Sigma)$
- $\lambda f.\lambda X.f(s(\{\sigma \in X \mid \texttt{true} \in e(\sigma)\})) \cup \{\sigma \in X \mid \texttt{false} \in e(\sigma)\}$ for some $e : \Sigma \to \mathcal{P}(\mathbb{V})$ and $s : \Sigma \to \mathcal{P}(\Sigma)$

Next Lecture

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- Trace Semantics

Jihyeok Park
jihyeok_park@korea.ac.kr
https://plrg.korea.ac.kr