

# Lecture 18 – Type Systems

## COSE212: Programming Languages

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2023 Fall

- We learned about **continuations** with the following topics:
  - **Continuations** (Lecture 14 & 15)
  - **First-Class Continuations** (Lecture 16)
  - **Compiling with continuations** (Lecture 17)
- From now on, we will learn about **type systems** with the following topics until the end of the semester:
  - Typed Languages
  - Typing Recursive Functions
  - Algebraic Data Types
  - Parametric Polymorphism
  - Subtype Polymorphism
  - Type Inference
- In this lecture, we will focus on the **motivation** and **basic concepts** of type systems.

## 1. Motivation: Safe Language Systems

- Detecting Run-Time Errors

- Dynamic vs Static Analysis

- Soundness vs Completeness

## 2. Type Systems

- Types

- Type Errors

- Type Checking

- Type Soundness

## 1. Motivation: Safe Language Systems

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Type Errors

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Type Soundness

So far, we have designed diverse programming languages with:

- **Syntax**: a grammar that defines the structure of programs
- **Semantics**: a set of rules that defines the meaning of programs

and implemented their **interpreters** in Scala:



However, we don't have any automatic system to **check** whether a program is evaluated without any **run-time errors**.

For example, following FAE expressions are syntactically correct, but they throw **run-time errors**:

```
/* FAE */  
x * 42           // error: free identifier  
0 + (x => x)     // error: cannot add a function  
1(2)           // error: cannot apply a number
```

Unexpected errors in **safety-critical software** cause serious problems:

 <p><b>June 4, 1996: Ariane-5 explodes after lift off</b> Today in History: June 4, 1996: Ariane-5 explodes after lift off Original Source: 2008-04 Abdul Samad, head of archive</p>	 <p><b>Knight Capital Says Trading Glitch Cost It</b> BY NATHANIEL POPPER AUGUST 2, 2012 6:07 AM 950 Runaway Trades Spread Turmoil Across Wall St.</p>	 <p><b>Heathrow Airport apologises for IT failure disruption</b> 14 February 2020</p>	 <p><b>Cruise recalls all its driverless cars after pedestrian hit and dragged</b> In another setback, Cruise updates software on 250 driverless cars to fix its 'Collision Data' By Steve Bannor Special November 6, 2020 12:14 PM • Updated November 6, 2020 at 2:08 PM</p>
<b>Rocket</b>	<b>Financial</b>	<b>Airport</b>	<b>Auto. Vehicle</b>
(1996)	(2012)	(2020)	(2023)

Then, how can we **prevent** such errors?

Can we **automatically** check whether a program does not have any **run-time errors**?

We can use various **analysis** techniques to detect run-time errors:



An **analyzer** is a program that takes a program as an input and determines whether the program has a certain property. In this case, the property is **run-time errors**.

We can categorize them into two groups:

- **Dynamic Analysis**: analyze programs by **executing** them
- **Static Analysis**: analyze programs **without executing** them

**Dynamic analysis** is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
}                         /* L6 */
```



**Dynamic analysis** is a program analysis technique by **executing** them.

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```

<b>L1</b>	-5
<b>L2</b>	-5
<b>L3</b>	5
<b>L4</b>	
<b>L5</b>	
<b>L6</b>	5

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```

<b>L1</b>	-5	42
<b>L2</b>	-5	
<b>L3</b>	5	
<b>L4</b>		42
<b>L5</b>		42
<b>L6</b>	5	42

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```

<b>L1</b>	-5	42	-7	99	0	...
<b>L2</b>	-5		-7			...
<b>L3</b>	5		7			...
<b>L4</b>		42		99	0	...
<b>L5</b>		42		99	0	...
<b>L6</b>	5	42	7	99	0	...

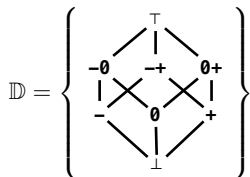
We can easily get the **behavior** of the program for each **single input**.

However, it is **difficult** to get all the **possible behaviors** of the program for **all the inputs**.

**Static analysis** is a program analysis technique **without executing** them.

Let's perform **static analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
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```



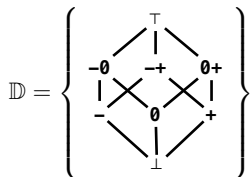
Let's define an **abstract domain**  $\mathbb{D}$  for integers to analyze the program.

$$\begin{array}{lll} \perp & = & \emptyset \\ 0 & = & \{0\} \\ -0 & = & - \cup 0 \\ \top & = & \mathbb{Z} \\ - & = & \{x \in \mathbb{Z} \mid x < 0\} \\ -+ & = & - \cup + \\ + & = & \{x \in \mathbb{Z} \mid x > 0\} \\ 0+ & = & 0 \cup + \end{array}$$

**Static analysis** is a program analysis technique **without executing** them.

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```



L1	
L2	
L3	
L4	
L5	
L6	

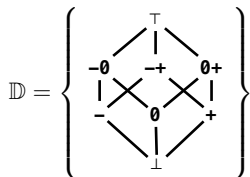
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<b>L1</b>	$\top$
<b>L2</b>	$-$
<b>L3</b>	$+$
<b>L4</b>	$0+$
<b>L5</b>	$0+$
<b>L6</b>	$0+$

Let's define an **abstract domain**  $\mathbb{D}$  for integers to analyze the program:

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We can prove that `abs` always returns a **non-negative** integer (i.e.,  $0+$ ).

- $\vdash \psi$  denotes that a statement  $\psi$  is **provable**.
- $\models \psi$  denotes that a statement  $\psi$  is **true**.

In a **sound** proof system, all **provable** statements are **true**.

$$\vdash \psi \quad \Longrightarrow \quad \models \psi$$

In a **complete** proof system, all **true** statements are **provable**.

$$\models \psi \quad \Longrightarrow \quad \vdash \psi$$

Analysis techniques can be used to prove that a program is **error-free**.

- $\vdash P$  denotes that a program  $P$  is **analyzed** as error-free.
- $\models P$  denotes that a program  $P$  is truly **error-free**.

Then, is dynamic/static analysis **sound** or **complete**?

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
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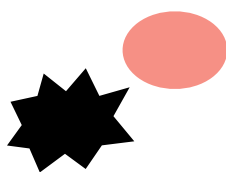
★ : Possible States    ● : Error States    ●●● : Dynamic Analysis    ● : Static Analysis



P<sub>1</sub>



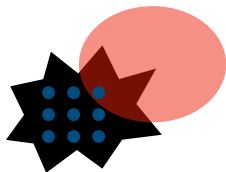
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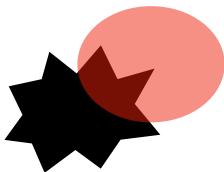
P<sub>3</sub>

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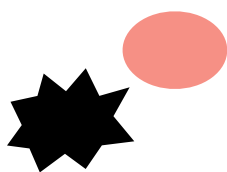
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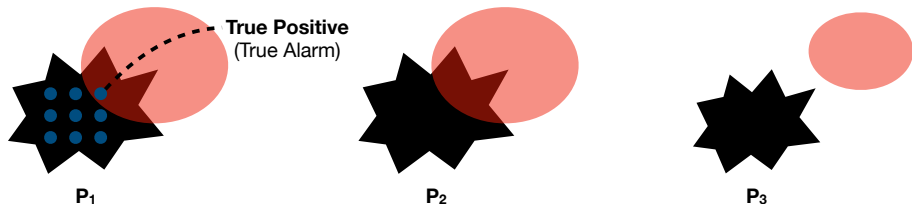
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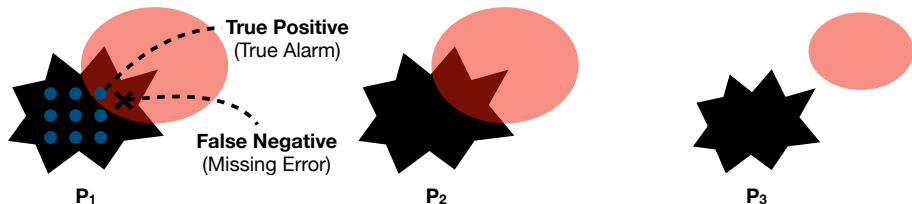
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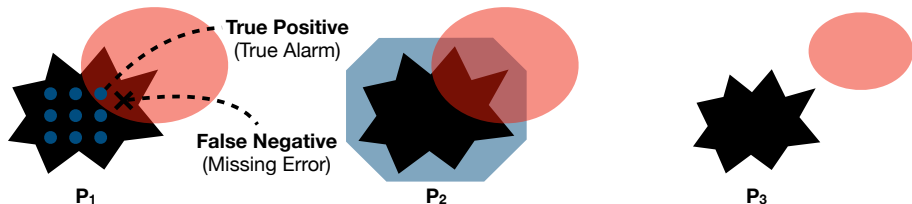
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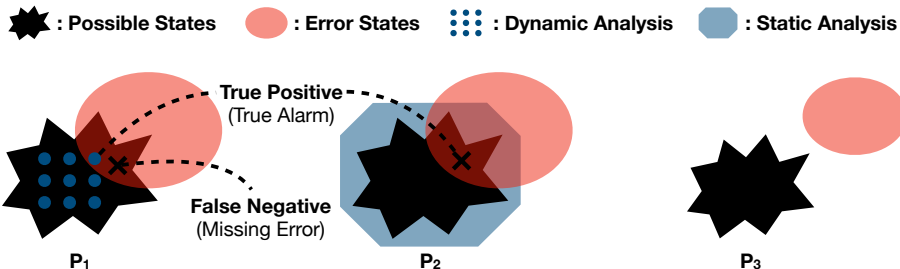


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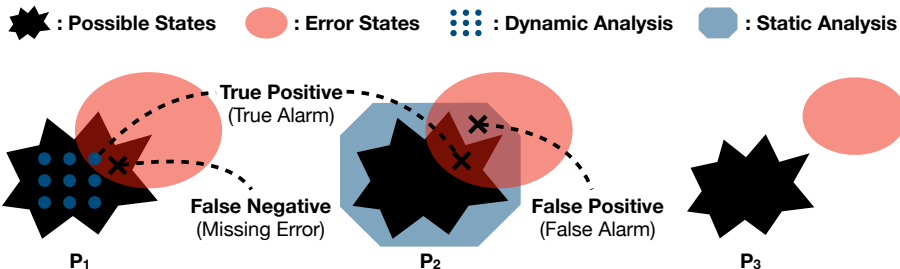
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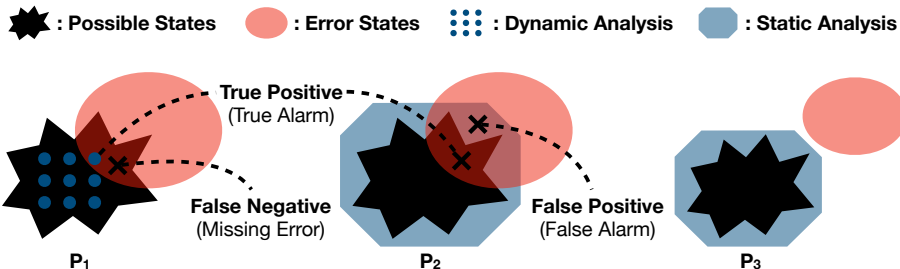
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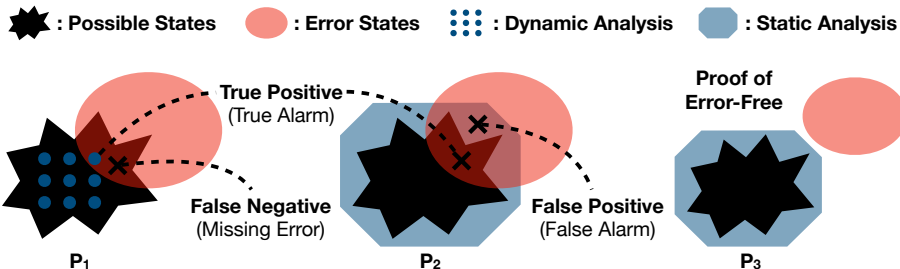


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Detecting Run-Time Errors

Dynamic vs Static Analysis

Soundness vs Completeness

## 2. Type Systems

Types

Type Errors

Type Checking

Type Soundness

## Definition (Types)

A **type** is a set of values.

For example, the `Int`, `Boolean`, and `Int => Int` types are defined as the following sets of values in Scala.

$$\text{Int} = \{n \in \mathbb{Z} \mid -2^{31} \leq n < 2^{31}\}$$

$$\text{Boolean} = \{\text{true}, \text{false}\}$$

$$\text{Int} \Rightarrow \text{Int} = \{f \mid f \text{ is a function from Int to Int}\}$$

```
val n: Int = 42           // 42   : Int
n + 1           // 43   : Int
val b: Boolean = n > 10  // true  : Boolean
def f(x: Int): Int = x + 1 // f    : Int => Int
f(42)           // 43   : Int
```

## Definition (Type Errors)

A **type error** occurs when a program tries to use a value having a type that is **incompatible** with the expected type.

For example, the following Scala program has type errors:

```
42 + true           // `Int` expected for `+`, but `Boolean` found
if (1) 2 else 3     // `Boolean` expected for `if`, but `Int` found
def f(x: Int): Int = x + 1
f(false)           // `Int` expected for `f`, but `Boolean` found
```

However, not all **run-time errors** are **type errors**:

```
42 / 0              // `ArithmeticException` at run-time
case class A(k: Int)
val x: A = null
x.k                 // `NullPointerException` at run-time
```

If the following conditions hold, we say “**the expression  $e$  has type  $\tau$** ”:

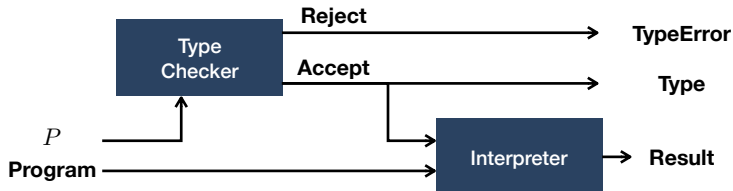
- $e$  does not cause any type error, and
- $e$  evaluates to a value of type  $\tau$  or does not terminate.

If so, we use the following notation and say that  $e$  is **well-typed**:

$$\boxed{\vdash e : \tau}$$

## Definition (Type Checking)

**Type checking** is a kind of static analysis checking whether a given expression  $e$  is **well-typed**. A **type checker** returns the **type** of  $e$  if it is well-typed, or rejects it and reports the detected **type error** otherwise.



## Definition (Type Soundness)

A **type system** is **sound** if it guarantees that a **well-typed** program will **never** cause a **type error** at run-time.

There are two categories of languages in the context of type system:

- **Statically-typed languages** (or simply typed-language) only allow **well-typed** programs to be executed.  
(e.g. Java, Scala, Haskell, OCaml, Rust, etc.)
- **Dynamically-typed languages** (or simply untyped-language) allow any program to be executed, and types exist only at run-time.  
(e.g. Python, Ruby, JavaScript, etc.)

Type systems in most statically-typed languages are designed to be **sound**.

## 1. Motivation: Safe Language Systems

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- Dynamic vs Static Analysis
- Soundness vs Completeness

## 2. Type Systems

- Types
- Type Errors
- Type Checking
- Type Soundness

# Homework #3

- Please see this document<sup>1</sup> on GitHub.
- The due date is Nov. 27 (Mon.).
- Please only submit `Implementation.scala` file to **Blackboard**.

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<sup>1</sup><https://github.com/ku-plrg-classroom/docs/tree/main/cose212/magnet>.



- Typed Languages

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