# Lecture 25 – Type Inference (1)

COSE212: Programming Languages

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PLRG

2023 Fall

# Recall

- **Polymorphism** is to use a single entity as **multiple types**, and there are various kinds of polymorphism:
  - **Parametric polymorphism**
  - **Subtype polymorphism**
  - **Ad-hoc polymorphism**
  - ...
- **PTFAE** – TFAE with **parametric polymorphism**.
- **STFAE** – TFAE with **subtype polymorphism**.

# Recall

- **Polymorphism** is to use a single entity as **multiple types**, and there are various kinds of polymorphism:
  - **Parametric polymorphism**
  - **Subtype polymorphism**
  - **Ad-hoc polymorphism**
  - ...
- **PTFAE** – TFAE with **parametric polymorphism**.
- **STFAE** – TFAE with **subtype polymorphism**.
- In this lecture, we will learn **type inference**.

# Type Inference

**Definition (Type Inference)**

**Type inference** is the process of automatically inferring the types of expressions.

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The goal of **type inference algorithm** is to infer the type of an expression without **explicit type annotations** given by programmers.

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The goal of **type inference algorithm** is to infer the type of an expression without **explicit type annotations** given by programmers.

Let's consider the following RFAE expression:

```
/* RFAE */
def sum(x) = if (x < 1) 0 else x + sum(x - 1)
sum
```

How can we **automatically infer** the type of `sum`?

# Type Inference

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**Type inference** is the process of automatically inferring the types of expressions.

The goal of **type inference algorithm** is to infer the type of an expression without **explicit type annotations** given by programmers.

Let's consider the following RFAE expression:

```
/* RFAE */
def sum(x) = if (x < 1) 0 else x + sum(x - 1)
sum
```

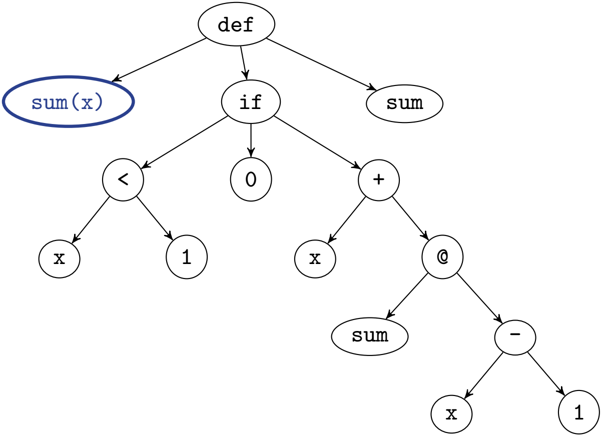
How can we **automatically infer** the type of `sum`?

1. Introduce **type variables** to denote unknown types
2. Collect the **type constraints** on the types
3. Find a **solution** (substitution of type variables) to the constraints

# Contents

# Contents

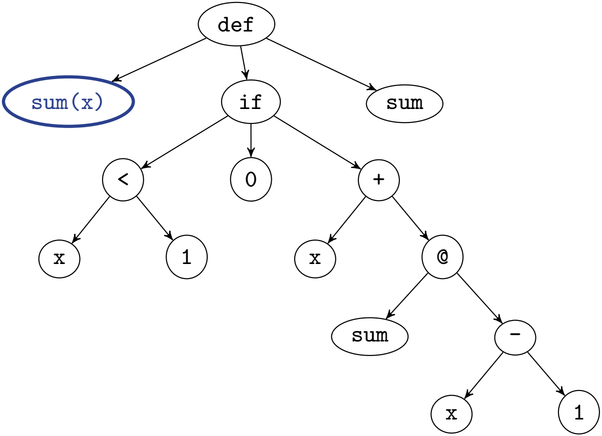
# Example 1 – sum




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | ??? |
| sum | ??? |

# Example 1 – sum




Type Environment

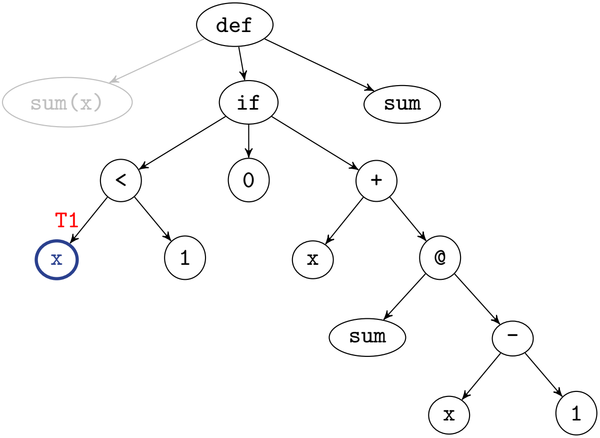
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |

Let's define **type variables** for unknown types.

# Example 1 – sum



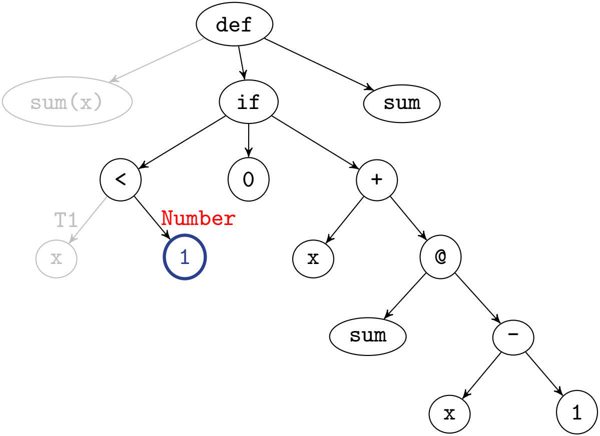

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |

# Example 1 – sum



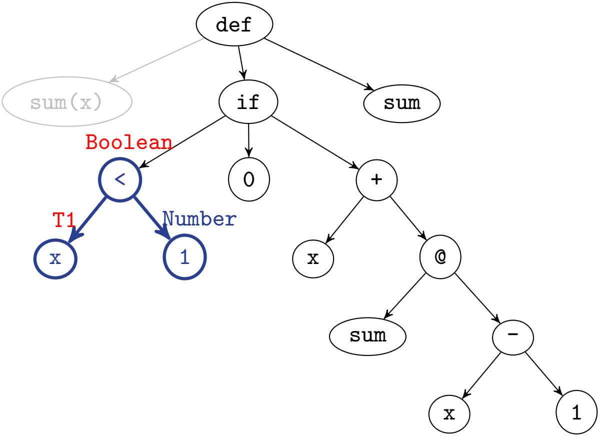

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |

# Example 1 – sum




Type Environment

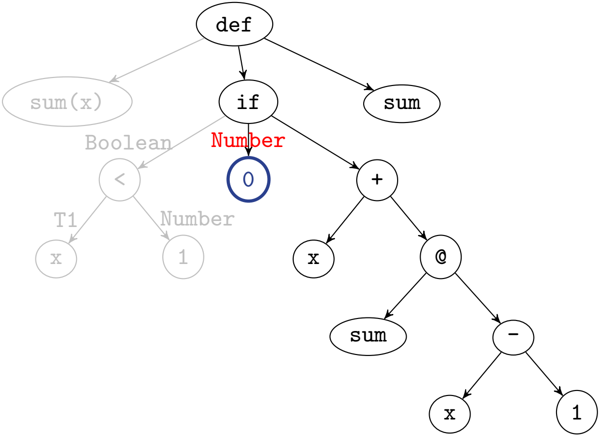
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

The **operands** of < must be of type **Number**.
So, we collected a **type constraint**: `T1 == Number`.

# Example 1 – sum



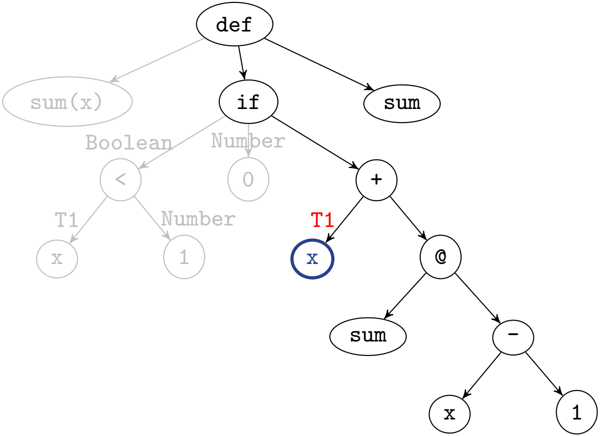

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

# Example 1 – sum



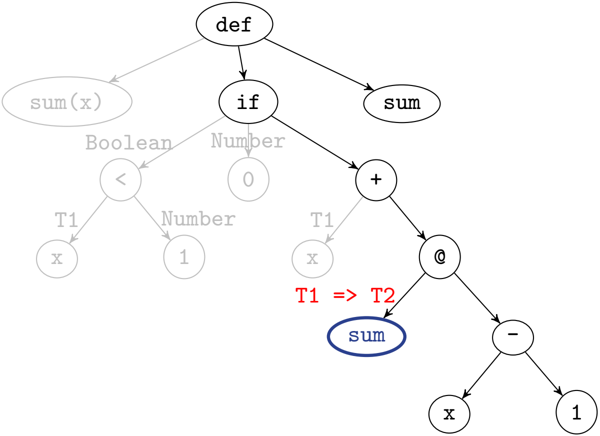

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

# Example 1 – sum



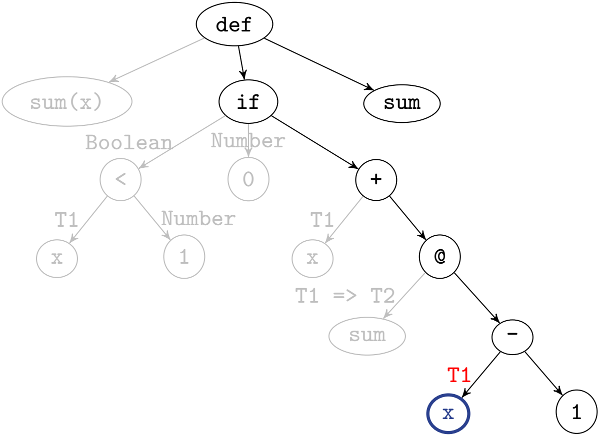

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

# Example 1 – sum



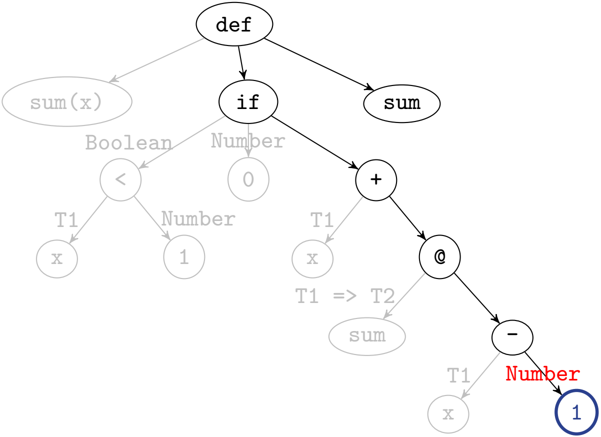

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

# Example 1 – sum



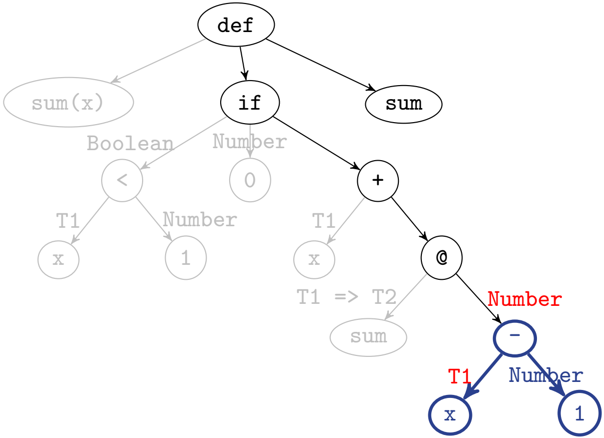

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

# Example 1 – sum




Type Environment

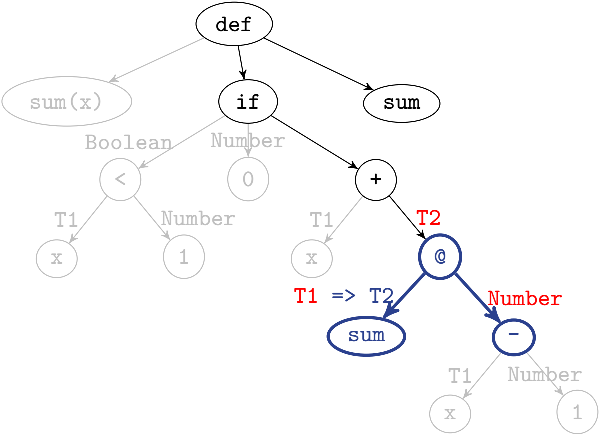
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

The **operands** of - must be of type **Number**.
We collected a **type constraint**: `T1 == Number`.
But, it is not a new constraint.

# Example 1 – sum




Type Environment

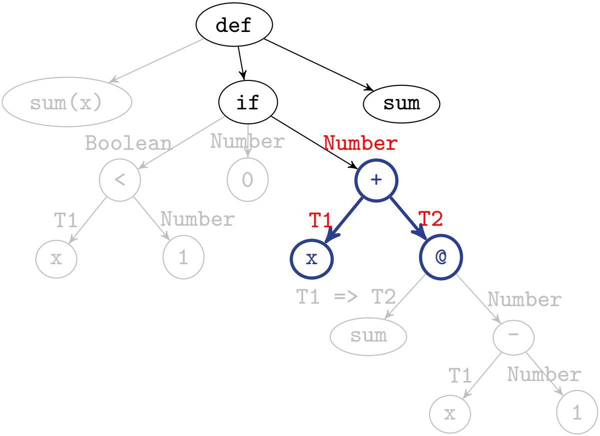
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | - |

The **argument type** should be equal to the **parameter type**.
We collected a **type constraint**: `T1 == Number`.
Again, it is not a new constraint.

# Example 1 – `sum`




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| `x` | `T1` |
| `sum` | `T1 => T2` |

Solution

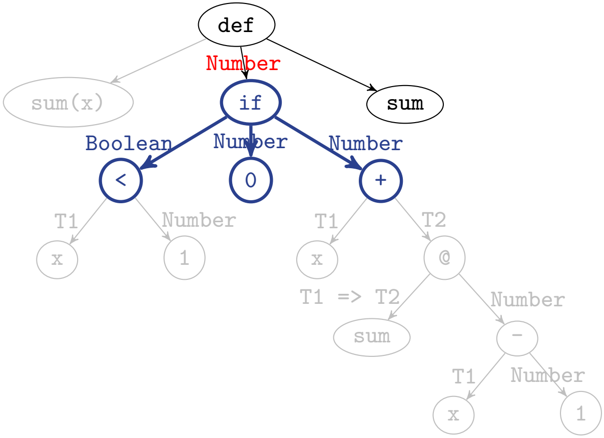
| $\mathbb{X}_{\alpha}$ | $\mathbb{T}$ |
|---|---|
| `T1` | `Number` |
| `T2` | `Number` |

The **operands** of + must be of type **Number**.

We collected **type constraints**: `T1 == Number` and `T2 == Number`.

The second one is a new constraint!

# Example 1 – sum



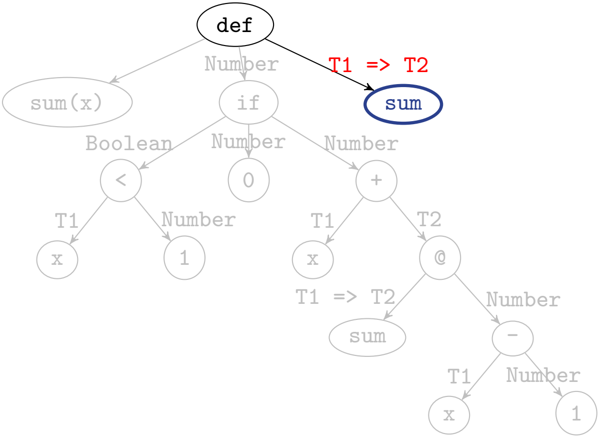

### Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

### Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | Number |

# Example 1 – sum




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| sum | T1 => T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | Number |

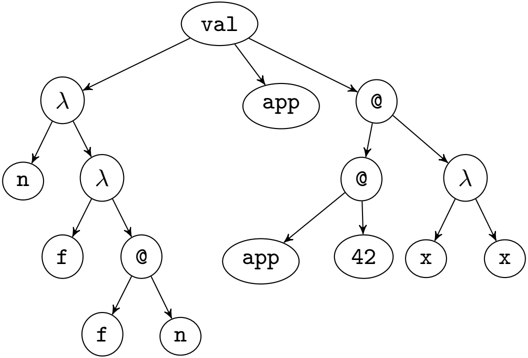
The type of `sum` is `T1 => T2`. Using the solution inferred by the collected constraints, we can instantiate it to `Number => Number`.

# Contents

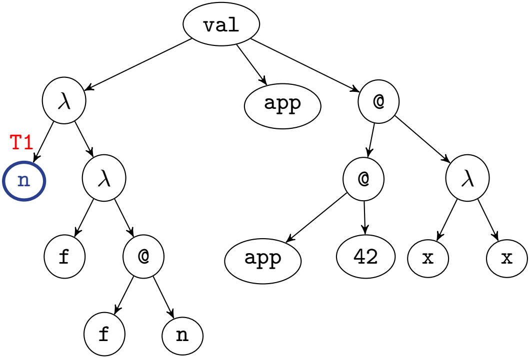
# Example 2 – app

Let's infer the type of the following RFAE expression:

```
/* RFAE */
val app = n => f => f(n)
app(42)(x => x)
```



# Example 2 – app




Type Environment

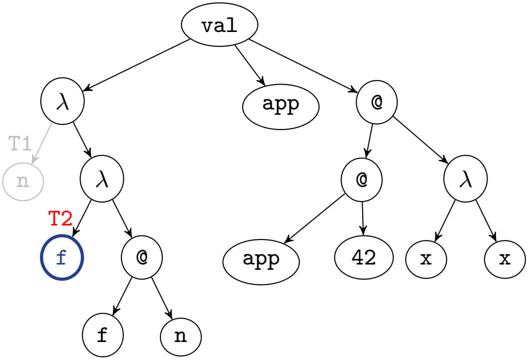
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| | |
| | |
| | |

Let's define a new **type variable T1** for the parameter n.

# Example 2 – app




Type Environment

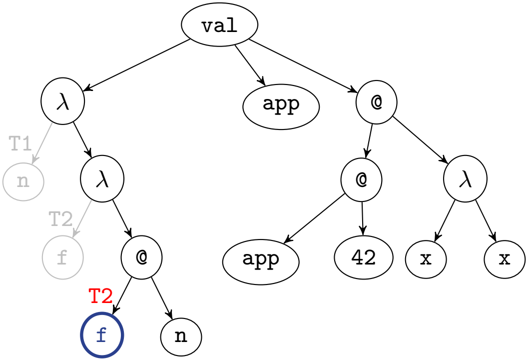
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| f | T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |
| | |
| | |

Let's define a new **type variable** T2 for the parameter `f`.

# Example 2 – app



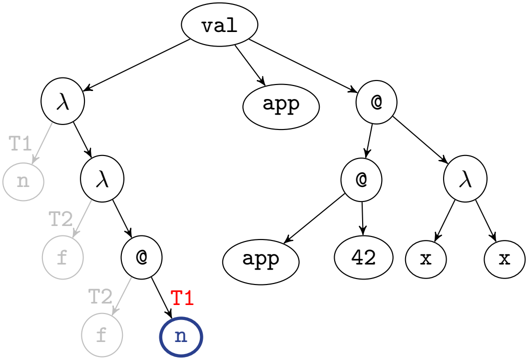

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| f | T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |
| | |
| | |

# Example 2 – app



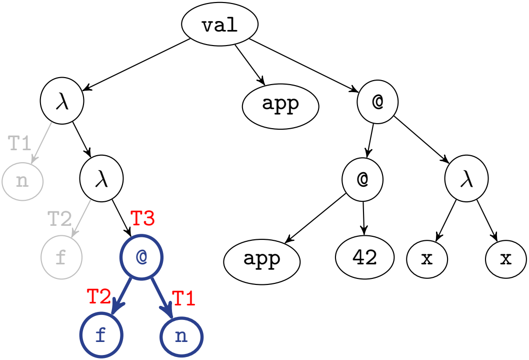

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| f | T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |
| | |
| | |

# Example 2 – app




Type Environment

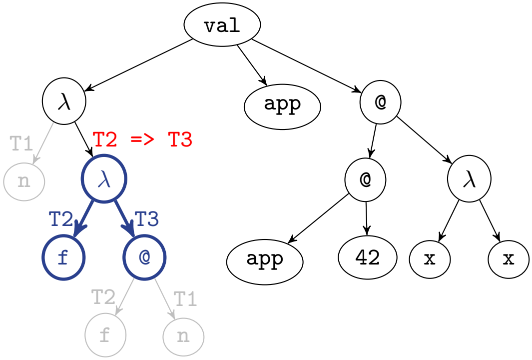
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| f | T2 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

The type T2 of f should be in the form of T1 => ???.
Let's define a new **type variable** T3 for ??? (the return type of f).
So, we collected a **type constraint**: T2 == T1 => T3.

# Example 2 – app



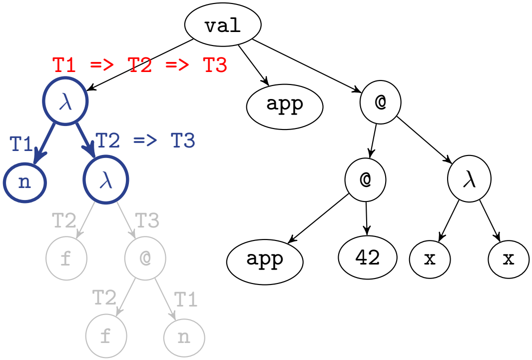

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| n | T1 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

# Example 2 – app



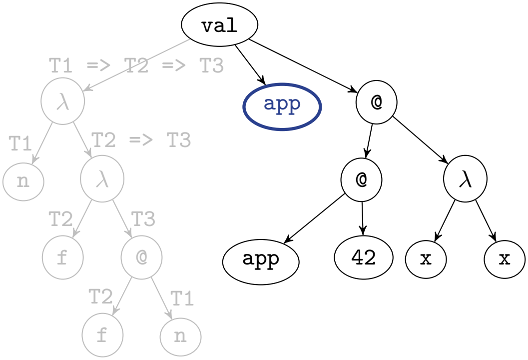

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

# Example 2 – app



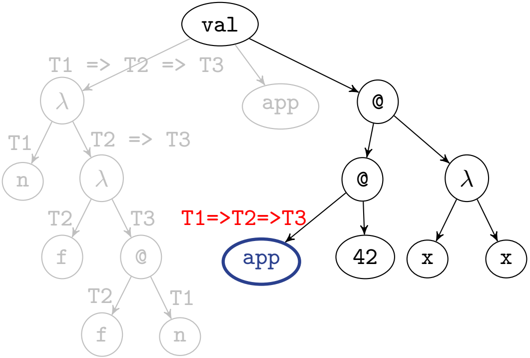

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

# Example 2 – app



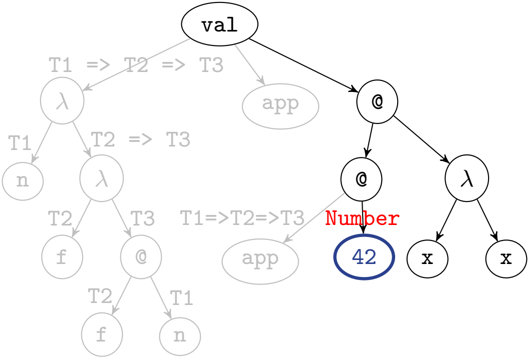

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

# Example 2 – app



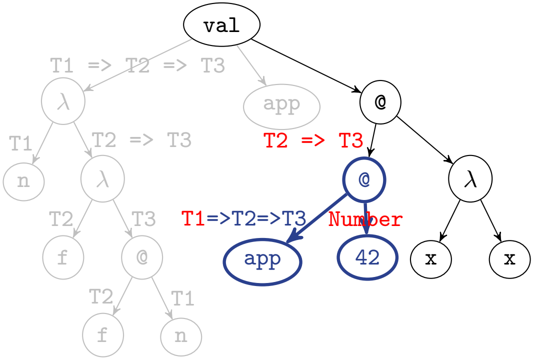

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | T1 => T3 |
| T3 | - |
| | |

# Example 2 – app




Type Environment

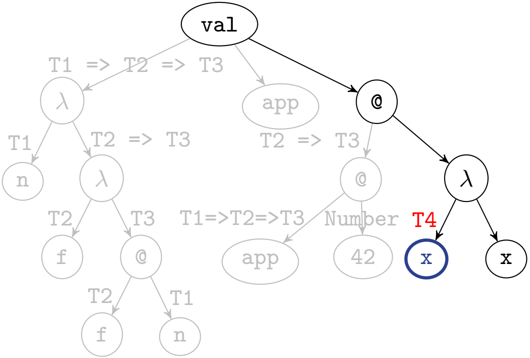
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | T1 => T3 |
| T3 | - |
| | |

The **parameter type** `T1` should be equal to the **argument type** `Number`.
So, we collected a **type constraint**: `T1 == Number`.

# Example 2 – app




Type Environment

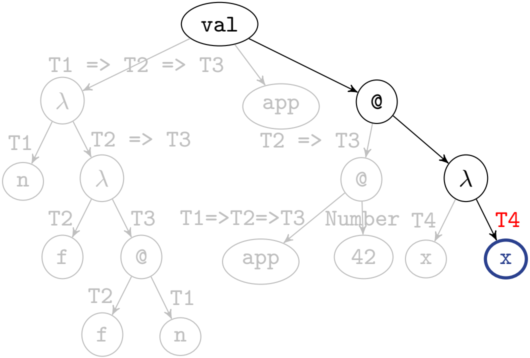
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| x | T4 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | T1 => T3 |
| T3 | - |
| T4 | - |

Let's define a new **type variable T4** for the parameter x.

# Example 2 – app



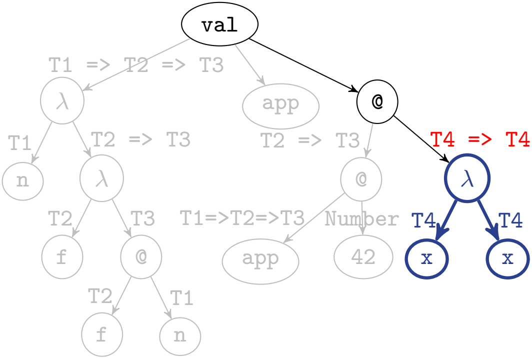

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| x | T4 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | T1 => T3 |
| T3 | - |
| T4 | - |

# Example 2 – app



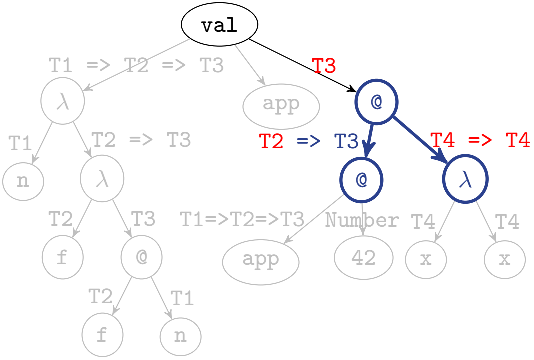

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | T1 => T2 => T3 |
| x | T4 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | Number |
| T2 | T1 => T3 |
| T3 | - |
| T4 | - |

# Example 2 – app




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| app | `T1 => T2 => T3` |
| x | `T4` |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | `Number` |
| T2 | `T1 => T3` |
| T3 | `Number` |
| T4 | `Number` |

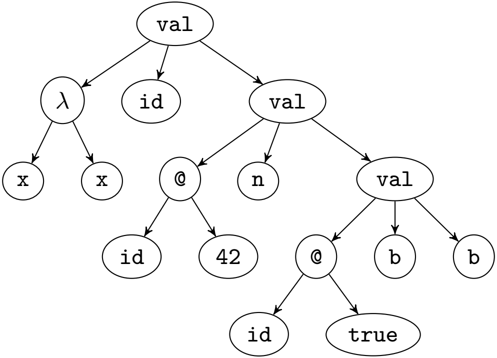
The **parameter type** `T2` should be equal to **argument type** `T4 => T4`.
We collected **type constraints**: `T3 == Number` and `T4 == Number`.
Finally, the entire expression has type `T3` (= `Number`).

# Contents

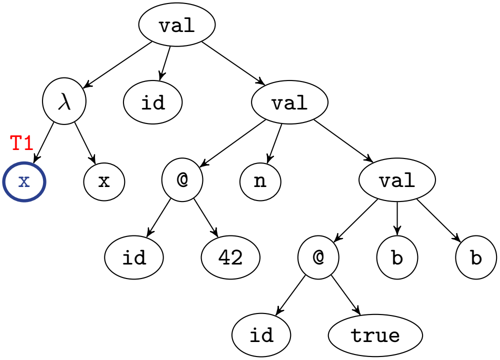
# Example 3 – `id`

Let's infer the type of the following RFAE expression:

```
/* RFAE */
val id = x => x
val n = id(42)
val b = id(true)
b
```



# Example 3 – `id`




Type Environment

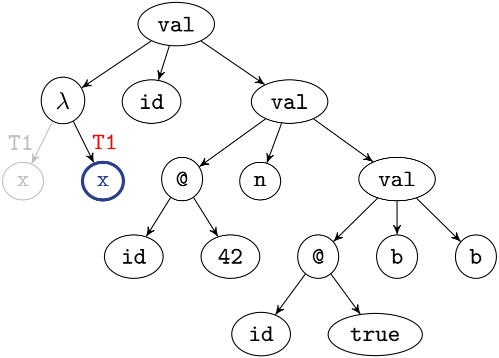
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| | |
| | |

Let's define a new **type variable T1** for the parameter `x`.

# Example 3 – id



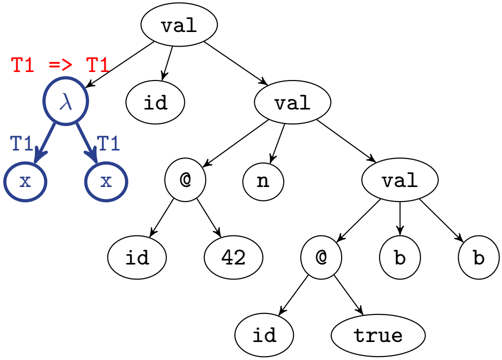

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| x | T1 |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| | |
| | |

# Example 3 – id



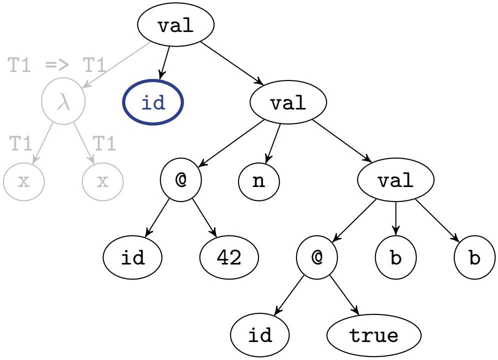

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| | |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| | |
| | |

# Example 3 – `id`




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| `id` | `[T1]` $\{$ `T1 => T1` $\}$ |
| | |
| | |

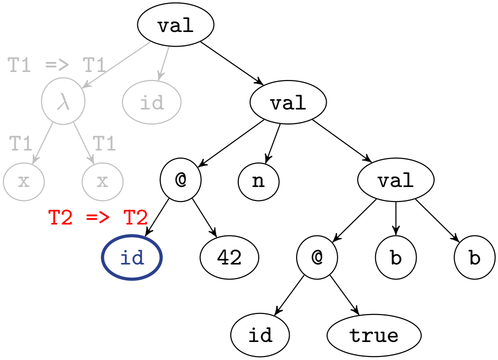
Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| `T1` | - |
| | |
| | |

Let's **generalize** the type `T1 => T1` into a **polymorphic type** for `id` with **type variable** `T1` as a **type parameter**.

We call this **let-polymorphism** because it only introduces polymorphism for the let-binding (e.g., `val`).

# Example 3 – `id`




Type Environment

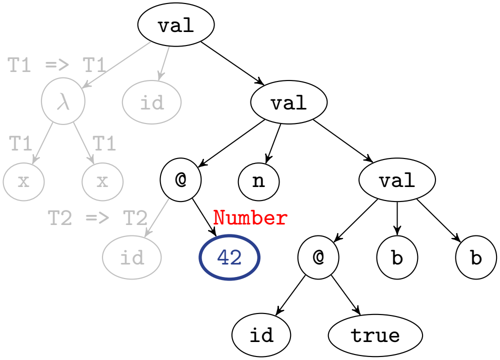
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |
| | |

Let's define a new **type variable `T2`** to **instantiate** the **type variable `T1`**. And, **substitute** `T1` with `T2`.

# Example 3 – id



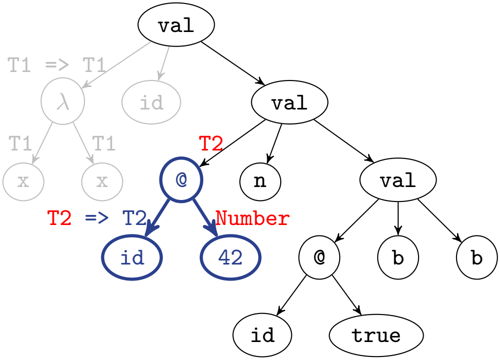

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | - |
| | |

# Example 3 – id




Type Environment

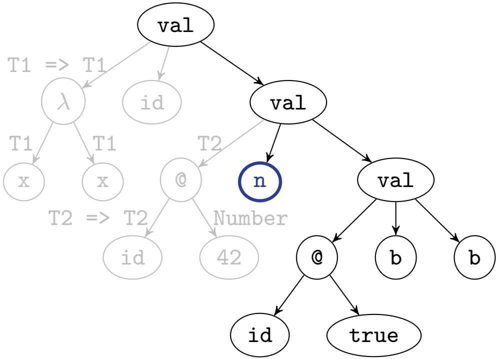
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| | |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | Number |
| | |

The **parameter type** T2 should be equal to **argument type** Number.
We collected a **type constraint**: T2 == Number.

# Example 3 – id




Type Environment

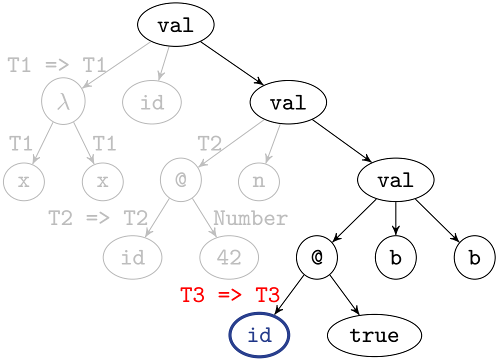
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | Number |
| | |

T2 is not a free type variable because it actually represents Number.
So, we will not introduce a polymorphic type in this case.

# Example 3 – id




Type Environment

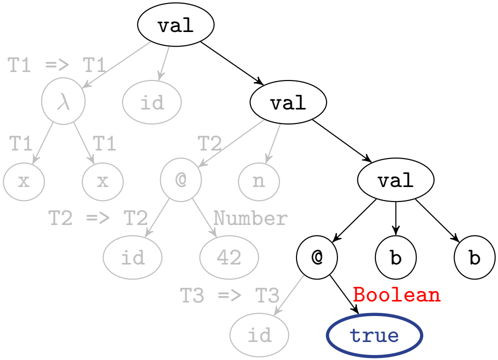
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | Number |
| T3 | - |

Let's define a new **type variable** `T3` to **instantiate** the **type variable** `T1`. And, **substitute** `T1` with `T3`.

# Example 3 – id



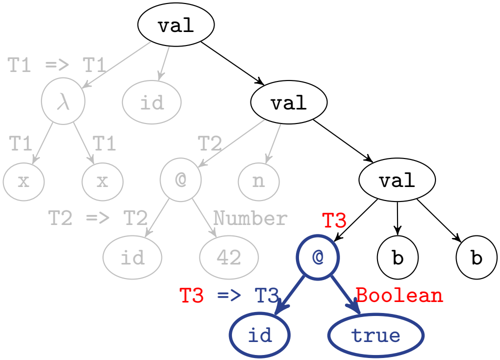

Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | Number |
| T3 | - |

# Example 3 – id




Type Environment

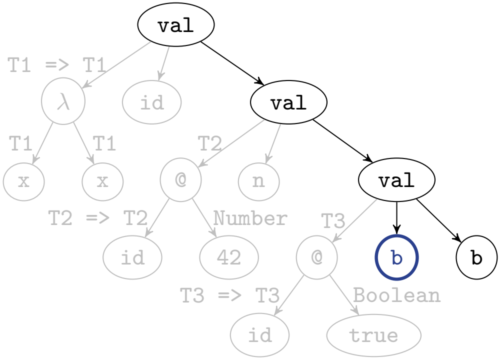
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| | |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | Number |
| T3 | Boolean |

The **parameter type** T3 should be equal to **argument type** Boolean.
We collected a **type constraint**: T3 == Boolean.

# Example 3 – id




Type Environment

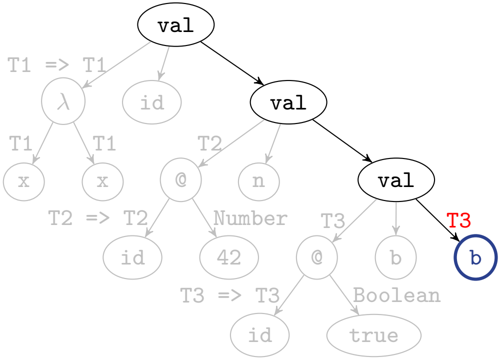
| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| b | T3 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | Number |
| T3 | Boolean |

T3 is not a free type variable because it actually represents Boolean.
So, we will not introduce a polymorphic type in this case.

# Example 3 – id




Type Environment

| $\mathbb{X}$ | $\mathbb{T}$ |
|---|---|
| id | [T1] { T1 => T1 } |
| n | T2 |
| b | T3 |

Solution

| $\mathbb{X}_\alpha$ | $\mathbb{T}$ |
|---|---|
| T1 | - |
| T2 | Number |
| T3 | Boolean |

Finally, the entire expression has type T3 (= Boolean).

# Summary

1. Example 1 – `sum`
2. Example 2 – `app`
3. Example 3 – `id`

# Next Lecture

- Type Inference (2)

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