

Lecture 7 – First-Class Functions

COSE212: Programming Languages

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2023 Fall

- **F1VAE – VAE with first-order functions**
 - Concrete and Abstract Syntax
 - Evaluation with Function Environments
 - Interpreters and Natural Semantics
 - Static Scoping vs Dynamic Scoping

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- **FVAE – VAE with first-class functions**
 - Concrete and Abstract Syntax
 - Interpreter and Natural Semantics

1. First-Class Functions

2. FVAE – VAE with First-Class Functions

Concrete Syntax

Abstract Syntax

3. Interpreter and Natural Semantics for FVAE

Closures – Functions as Values

Addition and Multiplication

Anonymous Functions

Function Application

Function Application (Dynamic Scoping)

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- ① **assigned** to a **variable**,
- ② **passed** as an **argument** to a function, and
- ③ **returned** from a function.

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- ② **passed** as an **argument** to a function, and
- ③ **returned** from a function.

For example, an integer is obviously a first-class citizen in Scala:

```
// 1. We can assign an integer to a variable.
```

```
val n: Int = 3
```

```
// 2. We can pass an integer as an argument to a function.
```

```
def square(n: Int): Int = n * n
```

```
square(3)    // 3 * 3 = 9
```

```
// 3. We can return an integer from a function.
```

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In Scala, **functions** are also **first-class citizens**, and we call them **first-class functions**.

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```
def inc(n: Int): Int = n + 1

// 1. We can assign a function to a variable.
val f: Int => Int = inc
val g: Int => Int = x => x + 1           // anonymous (lambda) function

// 2. We can pass a function as an argument to a function.
def twice(f: Int => Int, n: Int): Int = f(f(n))
twice(inc, 3)                         // inc(inc(3)) = 3 + 1 + 1 = 5
List(1, 2, 3).map(inc)                 // List(2, 3, 4)

// 3. We can return a function from a function.
def addN(n: Int): Int => Int = m => n + m
def addN(n: Int)(m: Int): Int = n + m // currying
addN(3)(5)                             // 3 + 5 = 8
val add3: Int => Int = addN(3)
add3(5)                                 // 3 + 5 = 8
```

Programming languages supporting **functional programming** paradigm treat functions as first-class citizens (i.e., **first-class functions**).

- Scala

```
List(1, 2, 3).map(x => x * 2) // List(2, 4, 6)
```

- Python

```
list(map(lambda x: x * 2, [1, 2, 3])) # [2, 4, 6]
```

- Rust

```
[1, 2, 3].iter().map(|x| x * 2).collect() // [2, 4, 6]
```

- Haskell

```
map (\x -> x * 2) [1, 2, 3] -- [2, 4, 6]
```

- ...

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Now, we want to extend VAE into FVAE with **first-class functions** rather than **first-order functions** in F1VAE.

```
/* FVAE */  
val addN = n => m => n + m;  
val add3 = addN(3);  
add3(5) // 3 + 5 = 8
```

```
/* FVAE */  
val inc = x => x + 1;  
val twice = f => n => f(f(n));  
twice(inc)(5) // 5 + 1 + 1 = 7
```

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- 1 **anonymous (lambda) functions**
- 2 **function applications**

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```
// expressions
<expr> ::= ...
        | <id> "=>" <expr>
        | <expr> "(" <expr> ")"
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Why not the following function application syntax?

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- 1 **anonymous (lambda) functions**
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Let's define the **concrete syntax** of FVAE in BNF:

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// expressions
<expr> ::= ...
        | <id> "=" <expr>
        | <expr> "(" <expr> ")"
```

Why not the following function application syntax?

```
| <id> "(" <expr> ")"
```

We cannot support **curried function applications** with the above syntax:

addN(3)(5)

Let's define the **abstract syntax** of FVAE in BNF:

Expressions $\mathbb{E} \ni e ::= \dots$

| | | |
|--|---------------|-------|
| | val x=e; e | (Val) |
| | x | (Id) |
| | $\lambda x.e$ | (Fun) |
| | $e(e)$ | (App) |

Let's define the **abstract syntax** of FVAE in BNF:

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| | val x=e; e | (Val) |
| | x | (Id) |
| | $\lambda x.e$ | (Fun) |
| | e(e) | (App) |

```
enum Expr:
  ...
  case Val(name: String, init: Expr, body: Expr)
  case Id(name: String)
  // anonymous (lambda) functions
  case Fun(param: String, body: Expr)
  // function applications
  case App(fun: Expr, arg: Expr)
```

For example, let's **parse** the following FVAE program:

```
/* FVAE */  
val addN = n => m => n + m;  
val add3 = addN(3);  
add3(5)
```

Then, the following **abstract syntax tree (AST)** is produced:

```
Val("addN",  
  Fun("n",  
    Fun("m",  
      Add(Id("n"), Id("m"))  
    )  
  ),  
  Val("add3",  
    App(Id("addN"), Const(3)),  
    App(Id("add3"), Const(5))  
  )  
)
```

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Let's evaluate the following FVAE program:

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How to evaluate the function applications $\text{addN}(3)$ and $\text{add3}(5)$?

$$\begin{aligned} [\text{addN} \mapsto v_0] & \quad \vdash \text{addN}(3) \Rightarrow v_1 \\ [\text{addN} \mapsto v_0, \text{add3} \mapsto v_1] & \quad \vdash \text{add3}(5) \Rightarrow v_2 \end{aligned}$$

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What's values of addN and add3 inside the environments?

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Functions! Let's define **values** as either **numbers** or **functions**:

$$\text{Values } \forall \ni v ::= n \mid \lambda x.e$$

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What's values of `addN` and `add3` inside the environments?

Functions! Let's define **values** as either **numbers** or **functions**:

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However, it is **NOT** what exactly we want to do. Why?

```
/* FVAE */  
val addN = n => m => n + m;  
val add3 = addN(3);  
add3(5) // 3 + 5 = 8
```

$[\text{addN} \mapsto v_0] \quad \vdash \text{addN}(3) \Rightarrow \lambda m.(n + m)$

$[\text{addN} \mapsto v_0, \text{add3} \mapsto v_1] \vdash \text{add3}(5) \Rightarrow v_2$

where $v_0 = \lambda n.\lambda m.(n + m)$ and $v_1 = \lambda m.(n + m)$.

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We know that m represents 5, but **what about** n ?

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We know that m represents 5, but **what about** n ?

Let's define **closures** as pairs of **functions** and its **environments**:

Values $\mathbb{V} \ni v ::= n \mid \langle \lambda x.e, \sigma \rangle$


```

/* FVAE */
val addN = n => m => n + m;
val add3 = addN(3);
add3(5) // 3 + 5 = 8
    
```

$$\begin{aligned}
 [\text{addN} \mapsto v_0] & \quad \vdash \text{addN}(3) \Rightarrow \lambda m.(n + m) \\
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 [\text{addN} \mapsto v_0, \text{add3} \mapsto v_1] & \quad \vdash \text{add3}(5) \Rightarrow 8
 \end{aligned}$$

where $v_0 = \langle \lambda n.\lambda m.(n + m), \emptyset \rangle$ and $v_1 = \langle \lambda m.(n + m), [n \mapsto 3] \rangle$.

```
// values
enum Value:
  case NumV(n: BigInt)
  case CloV(param: String, body: Expr, env: Env)
```

$$\begin{aligned} \text{Values } \mathbb{V} \ni v ::= n & \quad (\text{NumV}) \\ & \quad | \langle \lambda x.e, \sigma \rangle \quad (\text{CloV}) \end{aligned}$$

For FVAE, a **value** is either 1) a **number** n or 2) a **closure** $\langle \lambda x.e, \sigma \rangle$,

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enum Value:
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$$\begin{array}{l} \text{Values } \mathbb{V} \ni v ::= n \quad (\text{NumV}) \\ \quad \quad \quad | \langle \lambda x.e, \sigma \rangle \quad (\text{CloV}) \end{array}$$

For FVAE, a **value** is either 1) a **number** n or 2) a **closure** $\langle \lambda x.e, \sigma \rangle$,

```
// interpreter
def interp(expr: Expr, env: Env): Value = ???
```

$$\sigma \vdash e \Rightarrow v$$

and the interpreter takes an **expression** e with an **environment** σ and returns a **value** v (either a number or a closure):

For FVAE, we need to 1) implement the **interpreter**:

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For FVAE, we need to 1) implement the **interpreter**:

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and 2) define the **natural semantics** with environments:

$$\sigma \vdash e \Rightarrow v$$

| | | | | |
|-------------|--------------------|-------|---------------------------------------|--------|
| Expressions | $\mathbb{E} \ni e$ | $::=$ | ... | |
| | | | $\lambda x.e$ | (Fun) |
| | | | $e(e)$ | (App) |
| Values | $\mathbb{V} \ni v$ | $::=$ | n | (NumV) |
| | | | $\langle \lambda x.e, \sigma \rangle$ | (CloV) |

where

| | | | | |
|--------------|-------------------------|----------------------------|--------------|----------|
| Environments | $\sigma \in \mathbb{X}$ | $\xrightarrow{\text{fin}}$ | \mathbb{V} | (Env) |
| Integers | $n \in \mathbb{Z}$ | | | (BigInt) |
| Identifiers | $x \in \mathbb{X}$ | | | (String) |

```
def interp(expr: Expr, env: Env): Value = expr match
  ...

  case Add(l, r) => interp(l, env) + interp(r, env)

  case Mul(l, r) => interp(l, env) * interp(r, env)
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Add} \frac{\sigma \vdash e_1 \Rightarrow v_1 \quad \sigma \vdash e_2 \Rightarrow v_2}{\sigma \vdash e_1 + e_2 \Rightarrow v_1 + v_2}$$

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Is it correct?

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def interp(expr: Expr, env: Env): Value = expr match
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Is it correct? **No!**

We can only add or multiply **numbers** rather than arbitrary **values**.


```
def interp(expr: Expr, env: Env): Value = expr match
  ...

  case Add(l, r) => (interp(l, env), interp(r, env)) match
    case (NumV(l), NumV(r)) => NumV(l + r)
    case (l, r) => error(s"invalid operation: ${l.str} + ${r.str}")

  case Mul(l, r) => (interp(l, env), interp(r, env)) match
    case (NumV(l), NumV(r)) => NumV(l * r)
    case (l, r) => error(s"invalid operation: ${l.str} * ${r.str}")
```

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Let's refactor the code to avoid duplication using a helper function.

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Add(l, r) => (interp(l, env), interp(r, env)) match
    case (NumV(l), NumV(r)) => NumV(l + r)
    case (l, r) => error(s"invalid operation: ${l.str} + ${r.str}")
  case Mul(l, r) => (interp(l, env), interp(r, env)) match
    case (NumV(l), NumV(r)) => NumV(l * r)
    case (l, r) => error(s"invalid operation: ${l.str} * ${r.str}")
```

Let's define the following helper function:

```
type BOp = (BigInt, BigInt) => BigInt
// Type: (String, BOp) => ((Value, Value) => Value)
def numBOp(x: String, op: BOp): (Value, Value) => Value = ???
```

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Add(l, r) => (interp(l, env), interp(r, env)) match
    case (NumV(l), NumV(r)) => NumV(l + r)
    case (l, r) => error(s"invalid operation: ${l.str} + ${r.str}")
  case Mul(l, r) => (interp(l, env), interp(r, env)) match
    case (NumV(l), NumV(r)) => NumV(l * r)
    case (l, r) => error(s"invalid operation: ${l.str} * ${r.str}")
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```

Let's curry the helper function:

```
// Type: String => BOp => (Value, Value) => Value
def numBOp(x: String)(op: BOp)(l: Value, r: Value): Value = ???
```

```

type NBOp = (BigInt, BigInt) => BigInt
def numBOP(x: String)(op: NBOp)(l: Value, r: Value): Value = (l, r)
  match
    case (NumV(l), NumV(r)) => NumV(op(l, r))
    case (l, r) => error(s"invalid operation: ${l.str} $x ${r.str}")

val numAdd: (Value, Value) => Value = numBOP("+")(_ + _)
val numMul: (Value, Value) => Value = numBOP("*")(_ * _)

def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Add(l, r) => numAdd(interp(l, env), interp(r, env))
  case Mul(l, r) => numMul(interp(l, env), interp(r, env))
    
```

$$\sigma \vdash e \Rightarrow v$$

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```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Fun(p, b) => ???
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Fun} \frac{\text{???}}{\sigma \vdash \lambda x.e \Rightarrow \text{???}}$$

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Fun(p, b) => CloV(p, b, env)
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{Fun} \frac{}{\sigma \vdash \lambda x.e \Rightarrow \langle \lambda x.e, \sigma \rangle}$$

Construct a **closure** $\langle \lambda x.e, \sigma \rangle$ from the function $\lambda x.e$ with the current environment σ for **static scoping**.

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case App(f, e) => ???
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{App} \frac{\text{???}}{\sigma \vdash e_0(e_1) \Rightarrow \text{???}}$$


```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case App(f, e) => interp(f, env) match
    case CloV(p, b, fenv) => ...
    case v                => error(s"not a function: ${v.str}")
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{App} \frac{\sigma \vdash e_0 \Rightarrow \langle \lambda x. e_2, \sigma' \rangle \quad \dots}{\sigma \vdash e_0(e_1) \Rightarrow ???}$$

First, evaluate the **function expression** e_0 , check that it is a **closure**, and let $\langle \lambda x. e_2, \sigma' \rangle$ be the closure.

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case App(f, e) => interp(f, env) match
    case CloV(p, b, fenv) => ... interp(e, env) ...
    case v                => error(s"not a function: ${v.str}")
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{App} \frac{\sigma \vdash e_0 \Rightarrow \langle \lambda x. e_2, \sigma' \rangle \quad \sigma \vdash e_1 \Rightarrow v_1 \quad \dots}{\sigma \vdash e_0(e_1) \Rightarrow ???}$$

Then, evaluate the **argument expression** e_1 and let v_1 be the resulting **value**.

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case App(f, e) => interp(f, env) match
    case CloV(p, b, fenv) => interp(b, fenv + (p -> interp(e, env)))
    case v                => error(s"not a function: ${v.str}")
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{App} \frac{\sigma \vdash e_0 \Rightarrow \langle \lambda x. e_2, \sigma' \rangle \quad \sigma \vdash e_1 \Rightarrow v_1 \quad \sigma'[x \mapsto v_1] \vdash e_2 \Rightarrow v_2}{\sigma \vdash e_0(e_1) \Rightarrow v_2}$$

Finally, evaluate the **body expression** e_2 in the environment $\sigma'[x \mapsto v_1]$, where σ' denotes the **environment** captured at the **definition site** of the function for **static scoping**.

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case App(f, e) => interp(f, env) match
    case CloV(p, b, fenv) => interp(b, env + (p -> interp(e, env)))
    case v                => error(s"not a function: ${v.str}")
```

$$\sigma \vdash e \Rightarrow v$$

$$\text{App} \frac{\sigma \vdash e_0 \Rightarrow \langle \lambda x. e_2, \sigma' \rangle \quad \sigma \vdash e_1 \Rightarrow v_1 \quad \sigma[x \mapsto v_1] \vdash e_2 \Rightarrow v_2}{\sigma \vdash e_0(e_1) \Rightarrow v_2}$$

We can define **dynamic scoping** by using the current **environment** σ at the **call site** of the function instead of the environment σ' captured at the definition site of the function.

```
/* FVAE (static scoping) */
val x = 3;
val f = y => x * y;
val x = 4;
f(5) // 3 * 5 = 15
```

```
/* FVAE (dynamic scoping) */
val x = 3;
val f = y => x * y;
val x = 4;
f(5) // 4 * 5 = 20
```

$$\text{APP} \frac{\sigma_1 \vdash f \Rightarrow \langle \lambda y.(x \times y), \sigma_0 \rangle \quad \sigma_1 \vdash 5 \Rightarrow 5 \quad \sigma_0[y \mapsto 5] \vdash x \times y \Rightarrow 15}{\sigma_1 \vdash f(5) \Rightarrow 15}$$

$$\text{APP} \frac{\sigma_1 \vdash f \Rightarrow \langle \lambda y.(x \times y), \sigma_0 \rangle \quad \sigma_1 \vdash 5 \Rightarrow 5 \quad \sigma_1[y \mapsto 5] \vdash x \times y \Rightarrow 20}{\sigma_1 \vdash f(5) \Rightarrow 20}$$

where

$$\sigma_0 = [x \mapsto 3]$$

$$\sigma_1 = [x \mapsto 4, f \mapsto \langle \lambda y.(x \times y), \sigma_0 \rangle]$$

1. First-Class Functions

2. FVAE – VAE with First-Class Functions

Concrete Syntax

Abstract Syntax

3. Interpreter and Natural Semantics for FVAE

Closures – Functions as Values

Addition and Multiplication

Anonymous Functions

Function Application

Function Application (Dynamic Scoping)

- Please see this document¹ on GitHub.
 - Implement `interp` function.
 - Implement `interpDS` function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

¹<https://github.com/ku-plrg-classroom/docs/tree/main/cose212/fvae>.

- Lambda Calculus

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