Lecture 15 – Continuations (2)
COSE212: Programming Languages

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PLRG

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Recall

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- We will learn about **continuations** with the following topics:
  - **Continuations** (Lecture 14 & 15)
  - **First-Class Continuations** (Lecture 16)
  - **Compiling with continuations** (Lecture 17)

- A **continuation** represents the **rest of the computation**.
  - Continuation Passing Style (CPS)
  - Interpreter of FAE in CPS

- So far, we have defined **bit-step operational (natural) semantics** for our languages.

- In this lecture, we define **small-step operational (reduction) semantics** of FAE using **continuations** before moving on to **first-class continuations**.

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Contents

Recall: Interpreter of FAE in CPS

In the previous lecture, we represented the **continuation** of each expression in the interpreter of FAE as a **function** and implemented the interpreter in **continuation passing style** (CPS):

```
enum Value:
  case NumV(number: BigInt)
  case CloV(param: String, body: Expr, env: Env)
type Env = Map[String, Value]
type Cont = Value => Value

def interpCPS(expr: Expr, env: Env, k: Cont): Value = ...
```

Then, how can we define the **continuations** for the semantics of FAE?

$$
\begin{array}{llll}
\text{Values} & \mathbb{V} \ni v ::= n & & (\texttt{NumV}) \\
& \quad\quad\ | \ \langle \lambda x.e, \sigma \rangle & & (\texttt{CloV}) \\
\text{Environments} & \sigma \ \in \ \mathbb{X} \xrightarrow{\text{fin}} \mathbb{V} & & (\texttt{Env}) \\
\text{Continuations} & \quad = \ \texttt{???} & &
\end{array}
$$

Recall: Big-Step vs. Small-Step Semantics

The derivation of **big-step operational (natural) semantics** has a **tree** structure, where each derivation describes the **whole evaluation** of an expression:

$$\text{Mul } \dfrac{\text{Num } \dfrac{}{\varnothing \vdash 5 \Rightarrow 5} \quad \text{Add } \dfrac{\text{Num } \dfrac{}{\varnothing \vdash 1 \Rightarrow 1} \quad \text{Num } \dfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

but the derivation of **small-step operational (reduction) semantics** is **linear** and describes **each reduction step** of an expression:

$$5 * (1 + 2) \quad \rightarrow \quad 5 * 3 \quad \rightarrow \quad 15$$

It is possible but non-trivial to represent **continuations** in the **big-step operational (natural) semantics**.

Let's define the **small-step operational (reduction) semantics** of FAE to represent **continuations**.

Contents

$$\text{Mul} \dfrac{\text{Num} \dfrac{}{\varnothing \vdash 5 \Rightarrow 5} \quad \text{Add} \dfrac{\text{Num} \dfrac{}{\varnothing \vdash 1 \Rightarrow 1} \quad \text{Num} \dfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

Let's describe what happens **each step** of the evaluation of the expression $5 * (1 + 2)$ using **continuation** and a **value stack**:

$$\text{Mul} \dfrac{\text{Num} \dfrac{}{\varnothing \vdash 5 \Rightarrow 5} \quad \text{Add} \dfrac{\text{Num} \dfrac{}{\varnothing \vdash 1 \Rightarrow 1} \quad \text{Num} \dfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

Let's describe what happens **each step** of the evaluation of the expression $5 * (1 + 2)$ using **continuation** and a **value stack**:
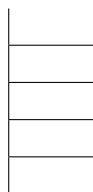
$\varnothing \vdash 5 * (1 + 2)$

Continuation

Value Stack

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$$\text{Mul} \cfrac{\text{Num} \cfrac{}{\varnothing \vdash 5 \Rightarrow 5} \quad \text{Add} \cfrac{\text{Num} \cfrac{}{\varnothing \vdash 1 \Rightarrow 1} \quad \text{Num} \cfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

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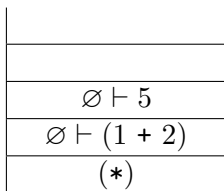
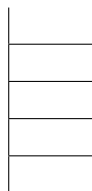| |
|---|
| |
| |
| $\varnothing \vdash 5$ |
| $\varnothing \vdash (1 + 2)$ |
| $(*)$ |

Continuation

Value Stack

Reduction Semantics of FAE

$$\text{Mul} \cfrac{\text{Num} \cfrac{}{\varnothing \vdash 5 \Rightarrow 5} \quad \text{Add} \cfrac{\text{Num} \cfrac{}{\varnothing \vdash 1 \Rightarrow 1} \quad \text{Num} \cfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

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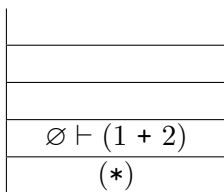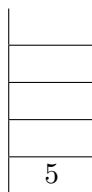
$\varnothing \vdash (1 + 2)$
$(*)$

Continuation

5

Value Stack

Reduction Semantics of FAE

$$\text{Mul} \cfrac{\text{Num} \cfrac{}{\varnothing \vdash 5 \Rightarrow 5} \quad \text{Add} \cfrac{\text{Num} \cfrac{}{\varnothing \vdash 1 \Rightarrow 1} \quad \text{Num} \cfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

Let's describe what happens **each step** of the evaluation of the expression
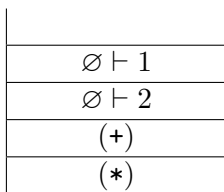$5 * (1 + 2)$ using **continuation** and a **value stack**:

$\varnothing \vdash 1$
$\varnothing \vdash 2$
$(+)$
$(*)$

Continuation

5

Value Stack

$$\text{Mul} \dfrac{\text{Num} \dfrac{}{\varnothing \vdash 5 \Rightarrow 5} \quad \text{Add} \dfrac{\text{Num} \dfrac{}{\varnothing \vdash 1 \Rightarrow 1} \quad \text{Num} \dfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

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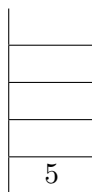
$\varnothing \vdash 2$
$(+)$
$(*)$

Continuation

1
5

Value Stack

$$\text{Mul} \dfrac{\text{Num} \dfrac{}{\varnothing \vdash 5 \Rightarrow 5} \quad \text{Add} \dfrac{\text{Num} \dfrac{}{\varnothing \vdash 1 \Rightarrow 1} \quad \text{Num} \dfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

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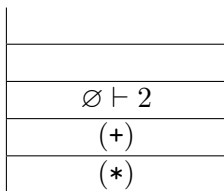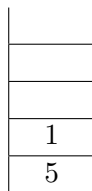
$(+)$
$(*)$

Continuation

2
1
5

Value Stack

Reduction Semantics of FAE

$$\text{MuL} \cfrac{\text{NuM} \cfrac{}{\varnothing \vdash 5 \Rightarrow 5} \quad \text{ADD} \cfrac{\text{NuM} \cfrac{}{\varnothing \vdash 1 \Rightarrow 1} \quad \text{NuM} \cfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

Let's describe what happens **each step** of the evaluation of the expression $5 * (1 + 2)$ using **continuation** and a **value stack**:
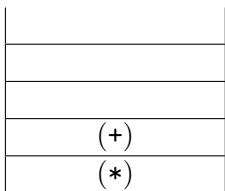

$(*)$	

Continuation

3	
5	

Value Stack

$$\text{Mul} \cfrac{\text{Num} \cfrac{}{\varnothing \vdash 5 \Rightarrow 5} \quad \text{Add} \cfrac{\text{Num} \cfrac{}{\varnothing \vdash 1 \Rightarrow 1} \quad \text{Num} \cfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

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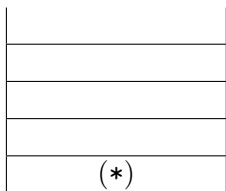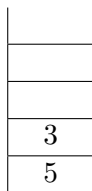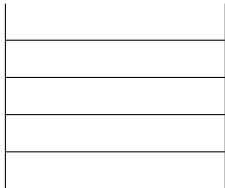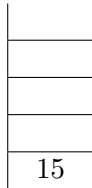

Continuation

Value Stack

OPLRG

- **Big-step operational (natural) semantics**:

$$\sigma \vdash e \Rightarrow v$$

- **Small-step operational (reduction) semantics**:

$$\langle \kappa \mid\mid s \rangle \to \langle \kappa \mid\mid s \rangle$$

where $\to \in (\mathbb{K} \times \mathbb{S}) \times (\mathbb{K} \times \mathbb{S})$ is a **reduction relation** between **states**.

$$
\begin{aligned}
\text{Continuations} \quad \mathbb{K} \ni \kappa ::= \;& \square \\
& \mid (\sigma \vdash e) :: \kappa \\
& \mid (+) :: \kappa \\
& \mid (*) :: \kappa \\
& \mid (@) :: \kappa
\end{aligned}
$$

$$\text{Value Stacks} \quad \mathbb{S} \ni s ::= \blacksquare \;\mid\; v :: s$$

```
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Num(n) => k(NumV(n))
```

$$\boxed{\langle \kappa \parallel s \rangle \rightarrow \langle \kappa \parallel s \rangle}$$

$$\text{Num} \quad \langle (\sigma \vdash n) :: \kappa \parallel s \rangle \quad \rightarrow \quad \langle \kappa \parallel n :: s \rangle$$

Addition

```
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    interpCPS(l, env, {
      lv => interpCPS(r, env, {
        rv => k(numAdd(lv, rv))
      })
    })
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$\text{Add}_1 \quad \langle (\sigma \vdash e_1 + e_2) :: \kappa \mid\mid s \rangle \quad \rightarrow \quad \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (\text{+}) :: \kappa \mid\mid s \rangle$

$\text{Add}_2 \quad \langle (\text{+}) :: \kappa \mid\mid n_2 :: n_1 :: s \rangle \quad \rightarrow \quad \langle \kappa \mid\mid (n_1 + n_2) :: s \rangle$

Multiplication

```
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Mul(l, r) =>
    interpCPS(l, env, {
      lv => interpCPS(r, env, {
        rv => k(numMul(lv, rv))
      })
    })
```

$$\langle \kappa \mid\mid s \rangle \to \langle \kappa \mid\mid s \rangle$$

$$\text{Mul}_1 \quad \langle (\sigma \vdash e_1 * e_2) :: \kappa \mid\mid s \rangle \quad \to \quad \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (*) :: \kappa \mid\mid s \rangle$$

$$\text{Mul}_2 \quad \langle (*) :: \kappa \mid\mid n_2 :: n_1 :: s \rangle \quad \to \quad \langle \kappa \mid\mid (n_1 \times n_2) :: s \rangle$$

```
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Id(x) => k(lookupId(x, env))
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$$\text{Id} \quad \langle (\sigma \vdash x) :: \kappa \mid\mid s \rangle \quad \rightarrow \quad \langle \kappa \mid\mid \sigma(x) :: s \rangle$$

```
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Fun(p, b) => k(CloV(p, b, env))
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$$\text{Fun} \quad \langle (\sigma \vdash \lambda x.e) :: \kappa \mid\mid s \rangle \quad \rightarrow \quad \langle \kappa \mid\mid \langle \lambda x.e, \sigma \rangle :: s \rangle$$

Function Application

```scala
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case App(f, e) => interpCPS(f, env, v => v match
    case CloV(p, b, fenv) =>
      interpCPS(e, env, v => {
        interpCPS(b, fenv + (p -> v), k)
      })
    case v => error(s"not a function: ${v.str}")
  )
```

$$\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle$$

$$\mathtt{App}_1 \langle (\sigma \vdash e_1(e_2)) :: \kappa \mid\mid s \rangle \rightarrow \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (@) :: \kappa \mid\mid s \rangle$$

$$\mathtt{App}_2 \langle (@) :: \kappa \mid\mid v_2 :: \langle \lambda x.e, \sigma \rangle :: s \rangle \rightarrow \langle (\sigma[x \mapsto v_2] \vdash e) :: \kappa \mid\mid s \rangle$$

Semantic Equivalence

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- The **reflexive transitive closure** $(\rightarrow^*)$ of $(\rightarrow)$:

$$\langle \kappa \mid\mid s \rangle \rightarrow^* \langle \kappa \mid\mid s \rangle$$

$$\frac{\langle \kappa \mid\mid s \rangle \rightarrow^* \langle \kappa' \mid\mid s' \rangle \quad \langle \kappa' \mid\mid s' \rangle \rightarrow \langle \kappa'' \mid\mid s'' \rangle}{\langle \kappa \mid\mid s \rangle \rightarrow^* \langle \kappa'' \mid\mid s'' \rangle}$$

- The **semantic equivalence** between natural and reduction semantics:

$$\varnothing \vdash e \Rightarrow v \qquad \Longleftrightarrow \qquad \langle (\varnothing \vdash e) :: \square \mid\mid \blacksquare \rangle \rightarrow^* \langle \square \mid\mid v :: \blacksquare \rangle$$

More generally, the following are equivalent:

$$\sigma \vdash e \Rightarrow v \qquad \Longleftrightarrow \qquad \langle (\sigma \vdash e) :: \kappa \mid\mid s \rangle \rightarrow^* \langle \kappa \mid\mid v :: s \rangle$$

for all $\sigma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{V}$, $e \in \mathbb{E}$, $v \in \mathbb{V}$, $\kappa \in \mathbb{K}$, and $s \in \mathbb{S}$.

Example

Let's interpret the expression $(\lambda x.(1 + \text{x}))(2)$:

$$
\begin{array}{ll}
& \langle\; (\varnothing \vdash (\lambda x.(1 + \text{x}))(2)) :: \square & \;||\; \blacksquare & \rangle \\
\overset{(\text{App}_1)}{\rightarrow} & \langle\; (\varnothing \vdash \lambda x.(1 + \text{x})) :: (\varnothing \vdash 2) :: (@) :: \square & \;||\; \blacksquare & \rangle \\
\overset{(\text{Fun})}{\rightarrow} & \langle\; (\varnothing \vdash 2) :: (@) :: \square & \;||\; \langle \lambda x.(1 + \text{x}), \varnothing \rangle :: \blacksquare & \rangle \\
\overset{(\text{Num})}{\rightarrow} & \langle\; (@) :: \square & \;||\; 2 :: \langle \lambda x.(1 + \text{x}), \varnothing \rangle :: \blacksquare & \rangle \\
\overset{(\text{App}_2)}{\rightarrow} & \langle\; ([x \mapsto 2] \vdash (1 + \text{x})) :: \square & \;||\; \blacksquare & \rangle \\
\overset{(\text{Add}_1)}{\rightarrow} & \langle\; ([x \mapsto 2] \vdash 1) :: ([x \mapsto 2] \vdash x) :: (+) :: \square & \;||\; \blacksquare & \rangle \\
\overset{(\text{Num})}{\rightarrow} & \langle\; ([x \mapsto 2] \vdash x) :: (+) :: \square & \;||\; 1 :: \blacksquare & \rangle \\
\overset{(\text{Id})}{\rightarrow} & \langle\; (+) :: \square & \;||\; 2 :: 1 :: \blacksquare & \rangle \\
\overset{(\text{Add}_2)}{\rightarrow} & \langle\; \square & \;||\; 3 :: \blacksquare & \rangle
\end{array}
$$

Thus, $\langle (\varnothing \vdash (\lambda x.(1 + \text{x}))(2)) :: \square \;||\; \blacksquare \rangle \rightarrow^* \langle \square \;||\; 3 :: \blacksquare \rangle$.

It means that the evaluation result of $(\lambda x.(1 + \text{x}))(2)$ is 3.

Contents

In our new implementation for FAE using CPS, we define **continuations** as **first-class functions** in Scala.

```scala
enum Value:
  case NumV(number: BigInt)
  case CloV(param: String, body: Expr, env: Env)
type Env = Map[String, Value]
type Cont = Value => Value

def interpCPS(expr: Expr, env: Env, k: Cont): Value = ...
```

How can we define continuations if we want to implement the interpreter for FAE in CPS using a **non-functional** language (e.g., C)?

Let's define the continuations as **data structures** (e.g., algebraic data types) in such languages.

We call them the **first-order representations of continuations**.

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```
enum Cont:
  case EmptyK
  case EvalK(env: Env, expr: Expr, k: Cont)
  case AddK(k: Cont)
  case MulK(k: Cont)
  case AppK(k: Cont)

type Stack = List[Value]
```

$$
\begin{aligned}
\text{Continuations} \quad \mathbb{K} \ni \kappa ::= &\ \square && (\texttt{EmptyK}) \\
| &\ (\sigma \vdash e) :: \kappa && (\texttt{EvalK}) \\
| &\ (\texttt{+}) :: \kappa && (\texttt{AddK}) \\
| &\ (\texttt{*}) :: \kappa && (\texttt{MulK}) \\
| &\ (\texttt{@}) :: \kappa && (\texttt{AppK})
\end{aligned}
$$

$$
\text{Value Stacks} \quad \mathbb{S} \ni s ::= \blacksquare \ \mid \ v :: s \quad (\texttt{List[Value]})
$$

We define a reduce function that takes a state $\langle \kappa \parallel s \rangle$ and **reduces** it to another state $\langle \kappa' \parallel s' \rangle$ using the reduction relation $\rightarrow$ we defined before:

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa' \parallel s' \rangle$$

```
def reduce(k: Cont, s: Stack): (Cont, Stack) = ???
```

And the evalK function **iteratively reduces** the state until it reaches the empty continuation $\square$ and returns the single value in the value stack:

```
def evalK(str: String): String =
  import Cont.*
  def aux(k: Cont, s: Stack): Value = reduce(k, s) match
    case (EmptyK, List(v)) => v
    case (k, s)            => aux(k, s)
  aux(EvalK(Map.empty, Expr(str), EmptyK), List.empty).str
```

$$\langle (\varnothing \vdash e) :: \square \parallel \blacksquare \rangle \rightarrow^* \langle \square \parallel v :: \blacksquare \rangle$$

First-Order Representations of Continuations

```
def reduce(k: Cont, s: Stack): (Cont, Stack) = (k, s) match
  case (EvalK(env, expr, k), s) => expr match
    ...
    case Add(l, r) => (EvalK(env, l, EvalK(env, r, AddK(k))), s)
  ...
  case (AddK(k), r :: l :: s) => (k, numAdd(l, r) :: s)
  ...
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$$\text{Add}_1 \quad \langle (\sigma \vdash e_1 + e_2) :: \kappa \mid\mid s \rangle \quad \rightarrow \quad \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (+) :: \kappa \mid\mid s \rangle$$

$$\text{Add}_2 \quad \langle (+) :: \kappa \mid\mid n_2 :: n_1 :: s \rangle \quad \rightarrow \quad \langle \kappa \mid\mid (n_1 + n_2) :: s \rangle$$

Similarly, we can define the reduce function for the other cases.

Summary

https://github.com/ku-plrg-classroom/docs/tree/main/cose212/fae-cps

- Please see above document on GitHub:
    - Implement `interpCPS` function.
    - Implement `reduce` function.

- It is just an exercise, and you **don't need to submit** anything.

- However, some exam questions might be related to this exercise.

- First-Class Continuations

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