Lecture 15 – Continuations (2)

COSE212: Programming Languages

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PLRG

2024 Fall

Recall

- We will learn about **continuations** with the following topics:
  - **Continuations** (Lecture 14 & 15)
  - **First-Class Continuations** (Lecture 16)
  - **Compiling with continuations** (Lecture 17)

- A **continuation** represents the **rest of the computation**.
  - Continuation Passing Style (CPS)
  - Interpreter of FAE in CPS

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- A **continuation** represents the **rest of the computation**.
  - Continuation Passing Style (CPS)
  - Interpreter of FAE in CPS
- So far, we have defined **bit-step operational (natural) semantics** for our languages.
- In this lecture, we define **small-step operational (reduction) semantics** of FAE using **continuations** before moving on to **first-class continuations**.

Contents

1. Recall
   - Recall: Interpreter of FAE in CPS
   - Recall: Big-Step vs. Small-Step Semantics
2. Reduction Semantics of FAE
   - Number
   - Addition
   - Multiplication
   - Identifier Lookup
   - Function Definition
   - Function Application
   - Semantic Equivalence
   - Example
3. First-Order Representations of Continuations

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Recall: Interpreter of FAE in CPS

In the previous lecture, we represented the **continuation** of each expression in the interpreter of FAE as a **function** and implemented the interpreter in **continuation passing style** (CPS):

```
enum Value:
  case NumV(number: BigInt)
  case CloV(param: String, body: Expr, env: Env)
type Env = Map[String, Value]
type Cont = Value => Value

def interpCPS(expr: Expr, env: Env, k: Cont): Value = ...
```

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In the previous lecture, we represented the **continuation** of each expression in the interpreter of FAE as a **function** and implemented the interpreter in **continuation passing style** (CPS):

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Then, how can we define the **continuations** for the semantics of FAE?

Values	$\mathbb{V} \ni v ::= n$	(NumV)
	$\mid \langle \lambda x.e, \sigma \rangle$	(CloV)
Environments	$\sigma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{V}$	(Env)
Continuations	$= ???$	

Recall: Big-Step vs. Small-Step Semantics

The derivation of **big-step operational (natural) semantics** has a **tree** structure, where each derivation describes the **whole evaluation** of an expression:

$$\textsc{Mul}\ \dfrac{\textsc{Num}\ \dfrac{}{\varnothing \vdash 5 \Rightarrow 5} \qquad \textsc{Add}\ \dfrac{\textsc{Num}\ \dfrac{}{\varnothing \vdash 1 \Rightarrow 1} \qquad \textsc{Num}\ \dfrac{}{\varnothing \vdash 2 \Rightarrow 2}}{\varnothing \vdash 1 + 2 \Rightarrow 3}}{\varnothing \vdash 5 * (1 + 2) \Rightarrow 15}$$

but the derivation of **small-step operational (reduction) semantics** is **linear** and describes **each reduction step** of an expression:

$$5 * (1 + 2) \quad \rightarrow \quad 5 * 3 \quad \rightarrow \quad 15$$

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It is possible but non-trivial to represent **continuations** in the **big-step operational (natural) semantics**.

Recall: Big-Step vs. Small-Step Semantics

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It is possible but non-trivial to represent **continuations** in the **big-step operational (natural) semantics**.

Let's define the **small-step operational (reduction) semantics** of FAE to represent **continuations**.

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Reduction Semantics of FAE

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Let's describe what happens **each step** of the evaluation of the expression $5 * (1 + 2)$ using **continuation** and a **value stack**:

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Value Stack

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$\varnothing \vdash 5$
$\varnothing \vdash (1 + 2)$
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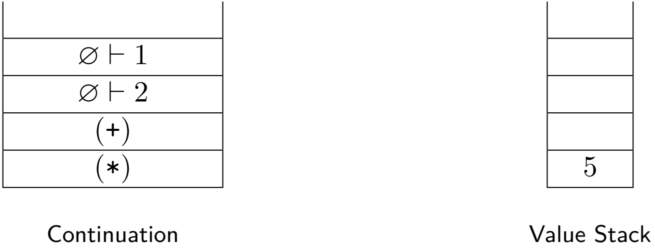

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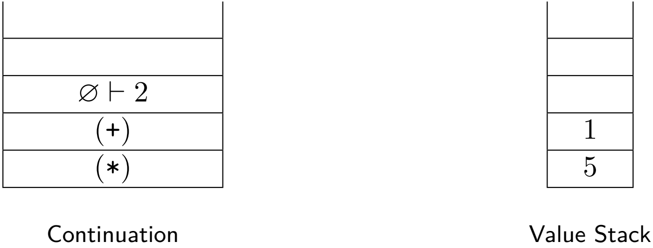

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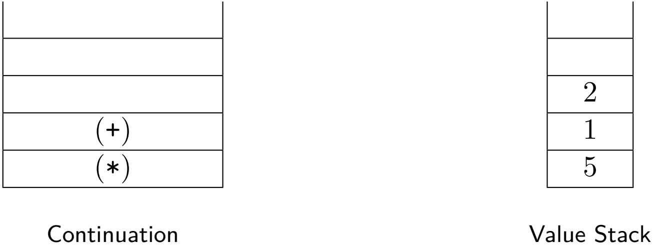

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Continuation

Value Stack

Reduction Semantics of FAE

- **Big-step operational (natural) semantics**:

$$\boxed{\sigma \vdash e \Rightarrow v}$$

- **Small-step operational (reduction) semantics**:

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

where $\rightarrow \in (\mathbb{K} \times \mathbb{S}) \times (\mathbb{K} \times \mathbb{S})$ is a **reduction relation** between **states**.

$$\begin{array}{llll} \text{Continuations} & \mathbb{K} \ni \kappa & ::= & \square \\ & & \mid & (\sigma \vdash e) :: \kappa \\ & & \mid & (\texttt{+}) :: \kappa \\ & & \mid & (\texttt{*}) :: \kappa \\ & & \mid & (\texttt{@}) :: \kappa \end{array}$$

$$\text{Value Stacks} \quad \mathbb{S} \ni s ::= \blacksquare \mid v :: s$$

Numbers

```
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Num(n) => k(NumV(n))
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$$\texttt{Num} \quad \langle (\sigma \vdash n) :: \kappa \mid\mid s \rangle \quad \rightarrow \quad \langle \kappa \mid\mid n :: s \rangle$$

Addition

```
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Add(l, r) =>
    interpCPS(l, env, {
      lv => interpCPS(r, env, {
        rv => k(numAdd(lv, rv))
      })
    })
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$$\texttt{Add}_1 \quad \langle (\sigma \vdash e_1 \texttt{ + } e_2) :: \kappa \mid\mid s \rangle \quad \rightarrow \quad \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (\texttt{+}) :: \kappa \mid\mid s \rangle$$

$$\texttt{Add}_2 \quad \langle (\texttt{+}) :: \kappa \mid\mid n_2 :: n_1 :: s \rangle \quad \rightarrow \quad \langle \kappa \mid\mid (n_1 + n_2) :: s \rangle$$

Multiplication

```scala
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Mul(l, r) =>
    interpCPS(l, env, {
      lv => interpCPS(r, env, {
        rv => k(numMul(lv, rv))
      })
    })
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$$\texttt{Mul}_1 \quad \langle (\sigma \vdash e_1 * e_2) :: \kappa \mid\mid s \rangle \quad \rightarrow \quad \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (*) :: \kappa \mid\mid s \rangle$$

$$\texttt{Mul}_2 \quad \langle (*) :: \kappa \mid\mid n_2 :: n_1 :: s \rangle \quad \rightarrow \quad \langle \kappa \mid\mid (n_1 \times n_2) :: s \rangle$$

Identifier Lookup

```
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Id(x) => k(lookupId(x, env))
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$$\texttt{Id} \quad \langle (\sigma \vdash x) :: \kappa \mid\mid s \rangle \quad \rightarrow \quad \langle \kappa \mid\mid \sigma(x) :: s \rangle$$

Function Definition

```
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case Fun(p, b) => k(CloV(p, b, env))
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$$\texttt{Fun} \quad \langle (\sigma \vdash \lambda x.e) :: \kappa \mid\mid s \rangle \quad \rightarrow \quad \langle \kappa \mid\mid \langle \lambda x.e, \sigma \rangle :: s \rangle$$

Function Application

```
def interpCPS(expr: Expr, env: Env, k: Cont): Value = expr match
  ...
  case App(f, e) => interpCPS(f, env, v => v match
    case CloV(p, b, fenv) =>
      interpCPS(e, env, v => {
        interpCPS(b, fenv + (p -> v), k)
      })
    case v => error(s"not a function: ${v.str}")
  )
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$$\texttt{App}_1 \langle (\sigma \vdash e_1(e_2)) :: \kappa \mid\mid s \rangle \rightarrow \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (\texttt{@}) :: \kappa \mid\mid s \rangle$$

$$\texttt{App}_2 \langle (\texttt{@}) :: \kappa \mid\mid v_2 :: \langle \lambda x.e, \sigma \rangle :: s \rangle \rightarrow \langle (\sigma[x \mapsto v_2] \vdash e) :: \kappa \mid\mid s \rangle$$

Semantic Equivalence

- The **reflexive transitive closure** $(\rightarrow^*)$ of $(\rightarrow)$:

$$\langle \kappa \mid\mid s \rangle \rightarrow^* \langle \kappa \mid\mid s \rangle$$

$$\frac{\langle \kappa \mid\mid s \rangle \rightarrow^* \langle \kappa' \mid\mid s' \rangle \qquad \langle \kappa' \mid\mid s' \rangle \rightarrow \langle \kappa'' \mid\mid s'' \rangle}{\langle \kappa \mid\mid s \rangle \rightarrow^* \langle \kappa'' \mid\mid s'' \rangle}$$

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- The **semantic equivalence** between natural and reduction semantics:

$$\varnothing \vdash e \Rightarrow v \qquad \iff \qquad \langle (\varnothing \vdash e) :: \square \mid\mid \blacksquare \rangle \to^* \langle \square \mid\mid v :: \blacksquare \rangle$$

More generally, the following are equivalent:

$$\sigma \vdash e \Rightarrow v \qquad \iff \qquad \langle (\sigma \vdash e) :: \kappa \mid\mid s \rangle \to^* \langle \kappa \mid\mid v :: s \rangle$$

for all $\sigma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{V}$, $e \in \mathbb{E}$, $v \in \mathbb{V}$, $\kappa \in \mathbb{K}$, and $s \in \mathbb{S}$.

Example

Let's interpret the expression $(\lambda x.(1 + \texttt{x}))(2)$:

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$$\langle\ (\varnothing \vdash (\lambda x.(1 + \mathtt{x}))(2)) :: \square \quad || \ \blacksquare \quad \rangle$$

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Let's interpret the expression $(\lambda x.(1 \texttt{ + } \texttt{x}))(2)$:

$$\begin{array}{rll} & \langle\ (\varnothing \vdash (\lambda x.(1 \texttt{ + } \texttt{x}))(2)) :: \square & \|\ \blacksquare \ \rangle \\ \xrightarrow{(\texttt{App}_1)} & \langle\ (\varnothing \vdash \lambda x.(1 \texttt{ + } \texttt{x})) :: (\varnothing \vdash 2) :: (\texttt{@}) :: \square & \|\ \blacksquare \ \rangle \end{array}$$

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\xrightarrow{(\mathtt{Fun})} & \langle\ (\varnothing \vdash 2) :: (\texttt{@}) :: \square & \|\ \langle \lambda x.(1 + \mathtt{x}), \varnothing \rangle :: \blacksquare & \rangle \\
\xrightarrow{(\mathtt{Num})} & \langle\ (\texttt{@}) :: \square & \|\ 2 :: \langle \lambda x.(1 + \mathtt{x}), \varnothing \rangle :: \blacksquare & \rangle \\
\xrightarrow{(\mathtt{App}_2)} & \langle\ ([x \mapsto 2] \vdash (1 + \mathtt{x})) :: \square & \|\ \blacksquare & \rangle \\
\xrightarrow{(\mathtt{Add}_1)} & \langle\ ([x \mapsto 2] \vdash 1) :: ([x \mapsto 2] \vdash x) :: (+) :: \square & \|\ \blacksquare & \rangle \\
\xrightarrow{(\mathtt{Num})} & \langle\ ([x \mapsto 2] \vdash x) :: (+) :: \square & \|\ 1 :: \blacksquare & \rangle \\
\xrightarrow{(\mathtt{Id})} & \langle\ (+) :: \square & \|\ 2 :: 1 :: \blacksquare & \rangle \\
\xrightarrow{(\mathtt{Add}_2)} & \langle\ \square & \|\ 3 :: \blacksquare & \rangle
\end{array}
$$

Example

Let's interpret the expression $(\lambda x.(1 \texttt{ + } \texttt{x}))(2)$:

$$
\begin{array}{rll}
 & \langle\ (\varnothing \vdash (\lambda x.(1 \texttt{ + } \texttt{x}))(2)) :: \square & \|\ \blacksquare \ \rangle \\
\xrightarrow{(\texttt{App}_1)} & \langle\ (\varnothing \vdash \lambda x.(1 \texttt{ + } \texttt{x})) :: (\varnothing \vdash 2) :: (\texttt{@}) :: \square & \|\ \blacksquare \ \rangle \\
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\xrightarrow{(\texttt{App}_2)} & \langle\ ([x \mapsto 2] \vdash (1 \texttt{ + } \texttt{x})) :: \square & \|\ \blacksquare \ \rangle \\
\xrightarrow{(\texttt{Add}_1)} & \langle\ ([x \mapsto 2] \vdash 1) :: ([x \mapsto 2] \vdash x) :: (\texttt{+}) :: \square & \|\ \blacksquare \ \rangle \\
\xrightarrow{(\texttt{Num})} & \langle\ ([x \mapsto 2] \vdash x) :: (\texttt{+}) :: \square & \|\ 1 :: \blacksquare \ \rangle \\
\xrightarrow{(\texttt{Id})} & \langle\ (\texttt{+}) :: \square & \|\ 2 :: 1 :: \blacksquare \ \rangle \\
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\end{array}
$$

Thus, $\langle(\varnothing \vdash (\lambda x.(1 \texttt{ + } \texttt{x}))(2)) :: \square \ \|\ \blacksquare\rangle \rightarrow^* \langle\square \ \|\ 3 :: \blacksquare\rangle$.

Example

Let's interpret the expression $(\lambda x.(1 + \texttt{x}))(2)$:

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\xrightarrow{(\texttt{Add}_2)} & \langle\ \square & \|\ 3 :: \blacksquare \ \rangle
\end{array}
$$

Thus, $\langle (\varnothing \vdash (\lambda x.(1 + \texttt{x}))(2)) :: \square \ \|\ \blacksquare \rangle \to^* \langle \square \ \|\ 3 :: \blacksquare \rangle$.

It means that the evaluation result of $(\lambda x.(1 + \texttt{x}))(2)$ is 3.

Contents

1. Recall
   - Recall: Interpreter of FAE in CPS
   - Recall: Big-Step vs. Small-Step Semantics

2. Reduction Semantics of FAE
   - Number
   - Addition
   - Multiplication
   - Identifier Lookup
   - Function Definition
   - Function Application
   - Semantic Equivalence
   - Example

3. First-Order Representations of Continuations

First-Order Representations of Continuations

In our new implementation for FAE using CPS, we define **continuations** as **first-class functions** in Scala.

```
enum Value:
  case NumV(number: BigInt)
  case CloV(param: String, body: Expr, env: Env)
type Env = Map[String, Value]
type Cont = Value => Value

def interpCPS(expr: Expr, env: Env, k: Cont): Value = ...
```

First-Order Representations of Continuations

PLRG

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How can we define continuations if we want to implement the interpreter for FAE in CPS using a **non-functional** language (e.g., C)?

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How can we define continuations if we want to implement the interpreter for FAE in CPS using a **non-functional** language (e.g., C)?

Let's define the continuations as **data structures** (e.g., algebraic data types) in such languages.

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How can we define continuations if we want to implement the interpreter for FAE in CPS using a **non-functional** language (e.g., C)?

Let's define the continuations as **data structures** (e.g., algebraic data types) in such languages.

We call them the **first-order representations of continuations**.

First-Order Representations of Continuations

```
enum Cont:
  case EmptyK
  case EvalK(env: Env, expr: Expr, k: Cont)
  case AddK(k: Cont)
  case MulK(k: Cont)
  case AppK(k: Cont)

type Stack = List[Value]
```

$$
\begin{array}{llll}
\text{Continuations} & \mathbb{K} \ni \kappa ::= \square & & (\texttt{EmptyK}) \\
& \quad\mid (\sigma \vdash e) :: \kappa & & (\texttt{EvalK}) \\
& \quad\mid (\texttt{+}) :: \kappa & & (\texttt{AddK}) \\
& \quad\mid (\texttt{*}) :: \kappa & & (\texttt{MulK}) \\
& \quad\mid (\texttt{@}) :: \kappa & & (\texttt{AppK}) \\
\\
\text{Value Stacks} & \mathbb{S} \ni s ::= \blacksquare \mid v :: s & & (\texttt{List[Value]})
\end{array}
$$

First-Order Representations of Continuations

We define a `reduce` function that takes a state $\langle \kappa \mid\mid s \rangle$ and **reduces** it to another state $\langle \kappa' \mid\mid s' \rangle$ using the reduction relation $\rightarrow$ we defined before:

$$\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa' \mid\mid s' \rangle$$

```
def reduce(k: Cont, s: Stack): (Cont, Stack) = ???
```

First-Order Representations of Continuations

We define a `reduce` function that takes a state $\langle \kappa \mid\mid s \rangle$ and **reduces** it to another state $\langle \kappa' \mid\mid s' \rangle$ using the reduction relation $\rightarrow$ we defined before:

$$\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa' \mid\mid s' \rangle$$

```
def reduce(k: Cont, s: Stack): (Cont, Stack) = ???
```

And the `evalK` function **iteratively reduces** the state until it reaches the empty continuation $\square$ and returns the single value in the value stack:

```
def evalK(str: String): String =
  import Cont.*
  def aux(k: Cont, s: Stack): Value = reduce(k, s) match
    case (EmptyK, List(v)) => v
    case (k, s)            => aux(k, s)
  aux(EvalK(Map.empty, Expr(str), EmptyK), List.empty).str
```

$$\langle (\varnothing \vdash e) :: \square \mid\mid \blacksquare \rangle \rightarrow^* \langle \square \mid\mid v :: \blacksquare \rangle$$

First-Order Representations of Continuations

```
def reduce(k: Cont, s: Stack): (Cont, Stack) = (k, s) match
  case (EvalK(env, expr, k), s) => expr match
    ...
    case Add(l, r) => (EvalK(env, l, EvalK(env, r, AddK(k))), s)
  ...
  case (AddK(k), r :: l :: s) => (k, numAdd(l, r) :: s)
  ...
```

$$\boxed{\langle \kappa \mid\mid s \rangle \rightarrow \langle \kappa \mid\mid s \rangle}$$

$$\texttt{Add}_1 \quad \langle (\sigma \vdash e_1 + e_2) :: \kappa \mid\mid s \rangle \quad \rightarrow \quad \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (+) :: \kappa \mid\mid s \rangle$$

$$\texttt{Add}_2 \quad \langle (+) :: \kappa \mid\mid n_2 :: n_1 :: s \rangle \quad \rightarrow \quad \langle \kappa \mid\mid (n_1 + n_2) :: s \rangle$$

Similarly, we can define the `reduce` function for the other cases.

Summary

1. Recall
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3. First-Order Representations of Continuations

Exercise #9

https://github.com/ku-plrg-classroom/docs/tree/main/cose212/fae-cps

- Please see above document on GitHub:
  - Implement `interpCPS` function.
  - Implement `reduce` function.

- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

Next Lecture

- First-Class Continuations

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