

Lecture 16 – First-Class Continuations

COSE212: Programming Languages

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- We will learn about **continuations** with the following topics:
 - **Continuations** (Lecture 14 & 15)
 - **First-Class Continuations** (Lecture 16)
 - **Compiling with continuations** (Lecture 17)
- A **continuation** represents the **rest of the computation**.
 - Continuation Passing Style (CPS)
 - Interpreter of FAE in CPS
 - Small-step operational (reduction) semantics of FAE
- In this lecture, we will learn **first-class continuations** and how to define the **control flow changes** in a program using them.
- **KFAE** – FAE with **first-class continuations**
 - Interpreter and Reduction semantics

1. First-Class Continuations

2. KFAE – FAE with First-Class Continuations Concrete/Abstract Syntax

3. Interpreter and Reduction Semantics for KFAE

Recall: Interpreter and Reduction Semantics for FAE

Interpreter and Reduction Semantics for KFAE

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In a programming language, an entity is said to be **first-class citizen** if it is treated as a **value**. In other words, it can be

- 1 **assigned** to a **variable**,
- 2 **passed** as an **argument** to a function, and
- 3 **returned** from a function.

For example, Scala supports **first-class functions**.

```
def inc(n: Int): Int = n + 1
// 1. We can assign a function to a variable.
val f: Int => Int = inc
// 2. We can pass a function as an argument to a function.
List(1, 2, 3).map(inc) // List(2, 3, 4)
// 3. We can return a function from a function.
def addN(n: Int): Int => Int = m => n + m
val add3: Int => Int = addN(3)
add3(5) // 3 + 5 = 8
```

Similarly, a **first-class continuation** means that a continuation is treated as a **value**. For example, Racket supports **first-class continuations**.

In Racket, we can 1) **capture** the continuation using `let/cc` and 2) **change** the program's **control flow** using the captured continuation.

For example, let's change the control flow of the following program:

```
; Racket
(* 2 (+ 3 5))
```

(Note that Racket uses the prefix notation (e.g., `(+ 1 2)`) instead of the infix notation (e.g., `1 + 2`).

by using the `let/cc` as follows:

```
; Racket
(* 2 (let/cc k (+ 3 (k 5)))) ; first-class continuation with `let/cc`
```

Let's compare the evaluation of the two expressions.

The original expression is evaluated in the following order:

```
; Racket  
(* 2 (+ 3 5))
```

- 1 Evaluate 2. (Result: 2)
- 2 Evaluate 3. (Result: 3)
- 3 Evaluate 5. (Result: 5)
- 4 Add the results of step 2 and 3. (Result: $3 + 5 = 8$)
- 5 Multiply the results of step 1 and 2 – 4. (Result: $2 * 8 = 16$)

What is the continuation of the expression $(+ 3 5)$? Step 5.

What is the continuation of the expression 5? Steps 4 and 5.

Let's 1) **capture** the continuation of $(+ 3 5)$ (i.e., 5) using `let/cc` and 2) **change** the **control flow** after evaluating 5 by using it as the continuation of 5 instead of the original one (i.e., 4 and 5).

We can change the program's control flow as follows:

```
; Racket  
(* 2 (let/cc k (+ 3 (k 5)))) ; first-class continuation with `let/cc`
```

- 1 Evaluate 2. (Result: 2)
- 2 Let k be the continuation of 2 – 7. (k is Step 8)
- 3 Evaluate 3. (Result: 3)
- 4 Evaluate k . (Result: Step 8)
- 5 Evaluate 5. (Result: 5)
- 6 Call the result of step 4 with that of 5. (**Replace Cont.**)
- 7 Add the results of step 3 and 4 – 6. (**Unreachable**)
- 8 Multiply the results of step 1 and 2 – 7. (Result: $2 * 5 = 10$)

It means that

- Step 2 defines the continuation of steps 2 – 7 as a value in k .
- Step 6 replaces the continuation of step 5 with k .

Some functional languages support **first-class continuations**.

- Racket

```
(* 2 (let/cc k (+ 3 (k 5)))) ; 2 * 5 = 10
```

- Ruby

```
2 * (callcc { |k| 3 + k.call(5)}) # 2 * 5 = 10
```

- Haskell

```
do
  x <- callCC $ \k -> do
    y <- k 5
    return $ 3 + y
  return $ 2 * x -- 2 * 5 = 10
```

- ...

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Now, let's extend FAE into KFAE with a new keyword `vcc` to capture the **first-class continuations**.

```
/* KFAE */  
2 * { vcc k; 3 + k(5) }
```

Here is another example of KFAE:

```
/* KFAE */  
{  
  vcc done;  
  val f = {  
    vcc exit;  
    2 * done(1 + {  
      vcc k;  
      exit(k)  
    })  
  };  
  f(3) * 5  
}
```

For KFAE, we need to extend **expressions** of FAE with

① **first-class continuations** (`vcc`)

We can extend the **concrete syntax** of FAE as follows:

```
// expressions  
<expr> ::= ... | "vcc" <id> ";" <expr>
```

and the **abstract syntax** of FAE as follows:

Expressions $\mathbb{E} \ni e ::= \dots \mid \text{vcc } x; e \quad (\text{Vcc})$

```
enum Expr:  
  ...  
  // first-class continuations  
  case Vcc(name: String, body: Expr)
```

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Then, what is the expected result of the following KFAE expressions?

```
/* KFAE */  
  
2 * { vcc k; 3 + k(5) }
```

Then, what is the expected result of the following KFAE expressions?

```
/* KFAE */  
// k is a continuation can be represented as `x => 2 * x`  
2 * { vcc k; 3 + k(5) }
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Then, what is the expected result of the following KFAE expressions?

```
/* KFAE */  
// k is a continuation can be represented as `x => 2 * x`  
2 * { vcc k; 3 + k(5) } // 2 * 5 = 10
```

```
/* KFAE */  
{  
  vcc done;  
  val f = {  
    vcc exit;  
    2 * done(1 + {  
      vcc k;  
      exit(k)  
    })  
  };  
  f(3) * 5  
}
```


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```
/* KFAE */  
// k is a continuation can be represented as `x => 2 * x`  
2 * { vcc k; 3 + k(5) } // 2 * 5 = 10
```

```
/* KFAE */  
{  
  vcc done;          // done = x => x  
  val f = {  
    vcc exit;  
    2 * done(1 + {  
      vcc k;  
      exit(k)  
    })  
  };  
  f(3) * 5  
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```

Then, what is the expected result of the following KFAE expressions?

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/* KFAE */
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2 * { vcc k; 3 + k(5) } // 2 * 5 = 10
```

```
/* KFAE */
{
  vcc done;          // done = x => x
  val f = {
    vcc exit;        // exit = y => val f = y; f(3) * 5
    2 * done(1 + {
      vcc k;
      exit(k)
    })
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  f(3) * 5
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Then, what is the expected result of the following KFAE expressions?

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/* KFAE */  
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```

```
/* KFAE */  
{  
  vcc done;          // done = x => x  
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    vcc exit;        // exit = y => val f = y; f(3) * 5  
    2 * done(1 + {  
      vcc k;          // k = z => val f = { 2 * done(1 + z) }; f(3) * 5  
      exit(k)  
    })  
  };  
  f(3) * 5  
}
```

Then, what is the expected result of the following KFAE expressions?

```
/* KFAE */
// k is a continuation can be represented as `x => 2 * x`
2 * { vcc k; 3 + k(5) } // 2 * 5 = 10
```

```
/* KFAE */
{
  vcc done;          // done = x => x
  val f = {
    vcc exit;       // exit = y => val f = y; f(3) * 5
    2 * done(1 + {
      vcc k;        // k = z => val f = { 2 * done(1 + z) }; f(3) * 5
      exit(k)
    })
  };
  f(3) * 5
}
```

// 4 (= 1 + 3)

In the previous lecture, we have defined the **first-order representation** of **continuations** with **value stack**:

```
enum Cont:
  case EmptyK
  case EvalK(env: Env, expr: Expr, k: Cont)
  case AddK(k: Cont)
  case MulK(k: Cont)
  case AppK(k: Cont)

type Stack = List[Value]
```

| | | |
|---------------|---|---------------|
| Continuations | $\mathbb{K} \ni \kappa ::= \square$ | (EmptyK) |
| | $(\sigma \vdash e) :: \kappa$ | (EvalK) |
| | $(+)$ $:: \kappa$ | (AddK) |
| | $(*)$ $:: \kappa$ | (MulK) |
| | $(@)$ $:: \kappa$ | (AppK) |
| Value Stacks | $\mathbb{S} \ni s ::= \blacksquare \mid v :: s$ | (List[Value]) |

Then, we have defined the **reduction relation** $\rightarrow \in (\mathbb{K} \times \mathbb{S}) \times (\mathbb{K} \times \mathbb{S})$ between **states** consisting of pairs of **continuations** and **value stacks**:

```
def reduce(k: Cont, s: Stack): (Cont, Stack) = ???
```

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa' \parallel s' \rangle$$

And the eval function **iteratively reduces** the state until it reaches the empty continuation \square and returns the single value in the value stack:

```
def eval(str: String): String =
  import Cont.*
  def aux(k: Cont, s: Stack): Value = reduce(k, s) match
    case (EmptyK, List(v)) => v
    case (k, s) => aux(k, s)
  aux(EvalK(Map.empty, Expr(str), EmptyK), List.empty).str
```

$$\langle (\emptyset \vdash e) :: \square \parallel \blacksquare \rangle \rightarrow^* \langle \square \parallel v :: \blacksquare \rangle$$

Now, let's extend the interpreter and reduction semantics for FAE to KFAE by adding the **first-class continuations**.

First, we need to extend the values of FAE with **continuation values** consisting of pairs of continuations and value stacks:

```
// values
enum Value:
  case NumV(number: BigInt)
  case CloV(param: String, body: Expr, env: Env)
  case ContV(cont: Cont, stack: Stack)
```

$$\text{Values } \forall \ni v ::= n \quad (\text{NumV})$$

$$| \langle \lambda x.e, \sigma \rangle \quad (\text{CloV})$$

$$| \langle \kappa \parallel s \rangle \quad (\text{ContV})$$

Then, let's fill out the missing cases in the reduce function and reduction rules for \rightarrow in the reduction semantics of KFAE.

```
def reduce(k: Cont, s: Stack): (Cont, Stack) = (k, s) match
  case (EvalK(env, expr, k), s) => expr match
    ...
    case Vcc(x, b) => (EvalK(env + (x -> ContV(k, s)), b, k), s)
```

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa \parallel s \rangle$$

$$\text{Vcc} \quad \langle (\sigma \vdash \text{vcc } x; e) :: \kappa \parallel s \rangle \rightarrow \langle (\sigma[x \mapsto \langle \kappa \parallel s \rangle] \vdash e) :: \kappa \parallel s \rangle$$

It defines a new immutable binding x in the environment σ that maps to a **continuation value** $\langle \kappa \parallel s \rangle$, and then evaluates the body expression e in the extended environment $\sigma[x \mapsto \langle \kappa \parallel s \rangle]$.


```

def reduce(k: Cont, s: Stack): (Cont, Stack) = (k, s) match
  case (EvalK(env, expr, k), s) => expr match
    ...
    case App(f, e) => (EvalK(env, f, EvalK(env, e, AppK(k))), s)
  ...
  case (AppK(k), a :: f :: s) => f match
    case CloV(p, b, fenv) => (EvalK(fenv + (p -> a), b, k), s)
    case ContV(k1, s1) => (k1, a :: s1)
    case v => error(s"not a function: ${v.str}")

```

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa \parallel s \rangle$$

$$\text{App}_1 \quad \langle (\sigma \vdash e_1(e_2)) :: \kappa \parallel s \rangle \rightarrow \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (@ :: \kappa \parallel s) \rangle$$

$$\text{App}_{2,\lambda} \quad \langle (@ :: \kappa \parallel v_2 :: \langle \lambda x.e, \sigma \rangle :: s \rangle \rightarrow \langle (\sigma[x \mapsto v_2] \vdash e) :: \kappa \parallel s \rangle$$

$$\text{App}_{2,\kappa} \quad \langle (@ :: \kappa \parallel v_2 :: \langle \kappa' \parallel s' \rangle :: s \rangle \rightarrow \langle \kappa' \parallel v_2 :: s' \rangle$$

The new $\text{App}_{2,\kappa}$ rule handles when the function expression evaluates to a continuation value $\langle \kappa' \parallel s' \rangle$. It changes the control flow to the continuation κ' with the given argument value v_2 and the value stack s' .

Example 1

Let's interpret the expression $2 * (\text{vcc } k; (3 + k(5)))$:

| | | | |
|---------------------------|--|---|-----------|
| | $\langle (\emptyset \vdash 2 * (\text{vcc } k; (3 + k(5)))) \rangle :: \square$ | $\parallel \blacksquare$ | \rangle |
| (Mul_1) | $\langle (\emptyset \vdash 2) \rangle :: \langle (\emptyset \vdash (\text{vcc } k; (3 + k(5)))) \rangle :: (*) \rangle :: \square$ | $\parallel \blacksquare$ | \rangle |
| \rightarrow | | | |
| (Num) | $\langle (\emptyset \vdash 2) \rangle :: \langle (\emptyset \vdash (\text{vcc } k; (3 + k(5)))) \rangle :: (*) \rangle :: \square$ | $\parallel 2 \rangle :: \blacksquare$ | \rangle |
| \rightarrow | | | |
| (Vcc) | $\langle (\sigma_0 \vdash (3 + k(5))) \rangle :: (*) \rangle :: \square$ | $\parallel 2 \rangle :: \blacksquare$ | \rangle |
| \rightarrow | | | |
| (Add_1) | $\langle (\sigma_0 \vdash 3) \rangle :: \langle (\sigma_0 \vdash k(5)) \rangle :: (+) \rangle :: (*) \rangle :: \square$ | $\parallel 2 \rangle :: \blacksquare$ | \rangle |
| \rightarrow | | | |
| (Num) | $\langle (\sigma_0 \vdash 3) \rangle :: \langle (\sigma_0 \vdash k(5)) \rangle :: (+) \rangle :: (*) \rangle :: \square$ | $\parallel 2 \rangle :: \blacksquare$ | \rangle |
| \rightarrow | | | |
| (Num) | $\langle (\sigma_0 \vdash k(5)) \rangle :: (+) \rangle :: (*) \rangle :: \square$ | $\parallel 3 \rangle :: 2 \rangle :: \blacksquare$ | \rangle |
| \rightarrow | | | |
| (App_1) | $\langle (\sigma_0 \vdash k) \rangle :: \langle (\sigma_0 \vdash 5) \rangle :: (@) \rangle :: (+) \rangle :: (*) \rangle :: \square$ | $\parallel 3 \rangle :: 2 \rangle :: \blacksquare$ | \rangle |
| \rightarrow | | | |
| (Id) | $\langle (\sigma_0 \vdash 5) \rangle :: (@) \rangle :: (+) \rangle :: (*) \rangle :: \square$ | $\parallel \langle \kappa_0 \parallel s_0 \rangle :: 3 \rangle :: 2 \rangle :: \blacksquare$ | \rangle |
| \rightarrow | | | |
| (Num) | $\langle (@) \rangle :: (+) \rangle :: (*) \rangle :: \square$ | $\parallel 5 \rangle :: \langle \kappa_0 \parallel s_0 \rangle :: 3 \rangle :: 2 \rangle :: \blacksquare$ | \rangle |
| \rightarrow | | | |
| $(\text{App}_{2,\kappa})$ | $\langle (*) \rangle :: \square$ | $\parallel 5 \rangle :: 2 \rangle :: \blacksquare$ | \rangle |
| \rightarrow | | | |
| (Mul_2) | $\langle \square \rangle$ | $\parallel 10 \rangle :: \blacksquare$ | \rangle |

$$\text{where } \begin{cases} \sigma_0 = [k \mapsto \langle \kappa_0 \parallel s_0 \rangle] \\ \kappa_0 = (*) \rangle :: \square \\ s_0 = 2 \rangle :: \blacksquare \end{cases}$$

Example 2

Let's interpret the expression $(\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)$:

$$\begin{array}{l}
 \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)) :: \square \rangle \quad \parallel \blacksquare \rangle \\
 \xrightarrow{(\text{App}_1)} \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))) :: (\emptyset \vdash 3) :: (@) :: \square \rangle \parallel \blacksquare \rangle \\
 \xrightarrow{(\text{Fun})} \langle (\emptyset \vdash 3) :: (@) :: \square \rangle \parallel \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 \xrightarrow{(\text{Num})} \langle (@) :: \square \rangle \parallel 3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 \xrightarrow{(\text{App}_{2,\lambda})} \langle (\sigma_0 \vdash \text{vcc } r; r(x + 1) * 2) :: \square \rangle \parallel \blacksquare \rangle \\
 \xrightarrow{(\text{Vcc})} \langle (\sigma_1 \vdash r(x + 1) * 2) :: \square \rangle \parallel \blacksquare \rangle \\
 \xrightarrow{(\text{Mul}_1)} \langle (\sigma_1 \vdash r(x + 1)) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle \parallel \blacksquare \rangle \\
 \xrightarrow{(\text{App}_1)} \langle (\sigma_1 \vdash r) :: (\sigma_1 \vdash x + 1) :: (@) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle \parallel \blacksquare \rangle \\
 \dots \\
 \xrightarrow{*} \langle (@) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle \parallel 4 :: \langle \square \parallel \blacksquare \rangle :: \blacksquare \rangle \\
 \xrightarrow{(\text{App}_{2,\kappa})} \langle \square \rangle \parallel 4 :: \blacksquare \rangle
 \end{array}$$

$$\text{where } \begin{cases} e_0 & = & \text{vcc } r; r(x + 1) * 2 \\ \sigma_0 & = & [x \mapsto 3] \\ \sigma_1 & = & \sigma_0[r \mapsto \langle \square \parallel \blacksquare \rangle] \end{cases}$$

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4. Control Statements

Many real-world programming languages support **control statements** to change the **control-flow** of a program.

For example, C++ supports **break**, **continue**, and **return** statements:

```
int sumEvenUntilZero(int xs[], int len) {
    if (len <= 0) return 0;           // directly return 0 if len <= 0
    int sum = 0;
    for (int i = 0; i < len; i++) {
        if (xs[i] == 0) break;       // stop the loop if xs[i] == 0
        if (xs[i] % 2 == 1) continue; // skip the rest if xs[i] is odd
        sum += xs[i];
    }
    return sum;                       // finally return the sum
}

int xs[] = {4, 1, 3, 2, 0, 6, 5, 8};
sumEvenUntilZero(xs, 8);           // 4 + 2 = 6
```

Let's represent them using **first-class continuations**!

- **return** statement:

```
x => body
```

means

```
x => { vcc return;  
      body // return(e) directly returns e to the caller  
}
```

- **break** and **continue** statements:

```
while (cond) body
```

means

```
{ vcc break;  
  while (cond) { vcc continue;  
    body // continue(e)/break(e) jumps to the next/end of the loop  
  }  
}
```

We can represent other control statements similarly, but think for yourself!

- exception in Python

```
try:
    x = y / z
except ZeroDivisionError:
    x = 0
```

- generator in JavaScript

```
const foo = function* () { yield 'a'; yield 'b'; yield 'c'; };
let str = '';
for (const c of foo()) { str = str + c; }
str // 'abc'
```

- coroutines in Kotlin
- async/await in C#
- ...

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4. Control Statements

<https://github.com/ku-plrg-classroom/docs/tree/main/cose212/kfae>

- Please see above document on GitHub:
 - Implement reduce function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise.

- Compiling with Continuations

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