Lecture 18 – Type Systems

COSE212: Programming Languages

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PLRG

2025 Fall

Recall

- We learned about **continuations** with the following topics:
  - **Continuations** (Lecture 14 & 15)
  - **First-Class Continuations** (Lecture 16)
  - **Compiling with continuations** (Lecture 17)

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  - Typed Languages
  - Typing Recursive Functions
  - Algebraic Data Types
  - Parametric Polymorphism
  - Subtype Polymorphism
  - Type Inference

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- In this lecture, we will focus on the **motivation** and **basic concepts** of type systems.

Contents

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Run-Time Errors

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- **Syntax**: a grammar that defines the structure of programs
- **Semantics**: a set of rules that defines the meaning of programs

and implemented their **interpreters** in Scala:

Run-Time Errors

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However, we don't have any automatic system to **check** whether a program is evaluated without any **run-time errors**.

Run-Time Errors

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- **Syntax**: a grammar that defines the structure of programs
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and implemented their **interpreters** in Scala:

However, we don't have any automatic system to **check** whether a program is evaluated without any **run-time errors**.

For example, following FAE expressions are syntactically correct, but they throw **run-time errors**:

```
/* FAE */
x * 42                 // error: free identifier
0 + (x => x)           // error: cannot add a function
1(2)                   // error: cannot apply a number
```

Errors in Saftety-Critical Software

Unexpected errors in **safety-critical software** cause serious problems:

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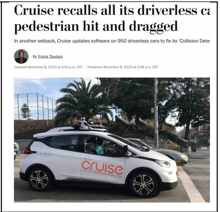
<td>
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			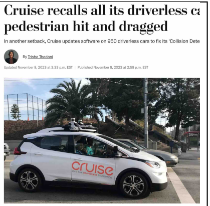
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Then, how can we **prevent** such errors?

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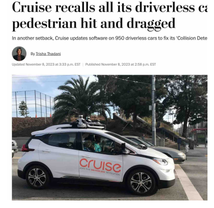
 June 4, 1996: Ariane-5 explodes after lift off Today in History: June 4, 1996: Ariane-5 explodes after lift off	 Knight Capital Says Trading Glitch Cost It BY NATHANIEL POPPER AUGUST 2, 2012 9:07 AM 356 Runaway Trades Spread Turmoil Across Wall St.	 Heathrow Airport apologises for IT failure disruption 16 February 2020 PA MEDIA Heathrow is the busiest airport in Europe	 Cruise recalls all its driverless ca pedestrian hit and dragged In another setback, Cruise updates software on 950 driverless cars to fix its 'Collision Dete
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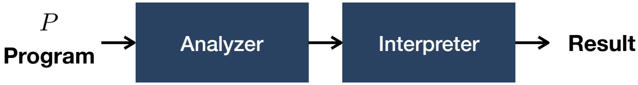
Then, how can we **prevent** such errors?

Can we **automatically** check whether a program does not have any **run-time errors**?

Detecting Run-Time Errors

We can use various **analysis** techniques to detect run-time errors:



An **analyzer** is a program that takes a program as an input and determines whether the program has a certain property. In this case, the property is **run-time errors**.

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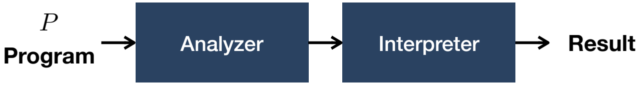

An **analyzer** is a program that takes a program as an input and determines whether the program has a certain property. In this case, the property is **run-time errors**.

We can categorize them into two groups:

- **Dynamic Analysis**: analyze programs by **executing** them
- **Static Analysis**: analyze programs **without executing** them

Dynamic Analysis

Dynamic analysis is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */
  if (x < 0)            /* L2 */
    -x                  /* L3 */
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L1	-5
L2	-5
L3	5
L4	
L5	
L6	5

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L1	-5	42
L2	-5	
L3	5	
L4		42
L5		42
L6	5	42

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L1	-5	42	-7	99	0	...
L2	-5		-7			...
L3	5		7			...
L4		42		99	0	...
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We can easily get the **behavior** of the program for each **single input**.

However, it is **difficult** to get all the **possible behaviors** of the program for **all the inputs**.

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Let's perform **static analysis** for the following Scala program:

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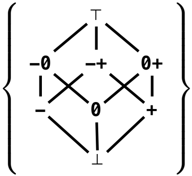
$\mathbb{D} =$



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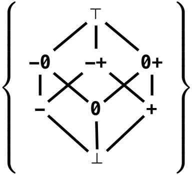
$\mathbb{D} =$ 

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$$
\begin{array}{lll}
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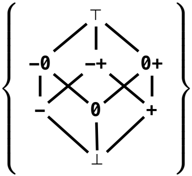
$\mathbb{D} =$ 

L1	
L2	
L3	
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L6	

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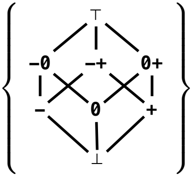
$\mathbb{D} =$ 

L1	$\top$
L2	-
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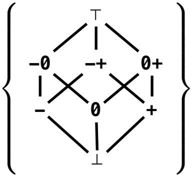
$\mathbb{D} =$ 

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We can prove that `abs` always returns a **non-negative** integer (i.e., 0+).

Soundness vs Completeness

- $\vdash \psi$ denotes that a statement ψ is **provable**.
- $\vDash \psi$ denotes that a statement ψ is **true**.

In a **sound** proof system, all **provable** statements are **true**.

$$\vdash \psi \qquad \Longrightarrow \qquad \vDash \psi$$

In a **complete** proof system, all **true** statements are **provable**.

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Analysis techniques can be used to prove that a program is **error-free**.

- $\vdash P$ denotes that a program P is **analyzed** as error-free.
- $\vDash P$ denotes that a program P is truly **error-free**.

Then, is dynamic/static analysis **sound** or **complete**?

Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
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  - It suffers from **missing errors** (**false negative (FN)**).

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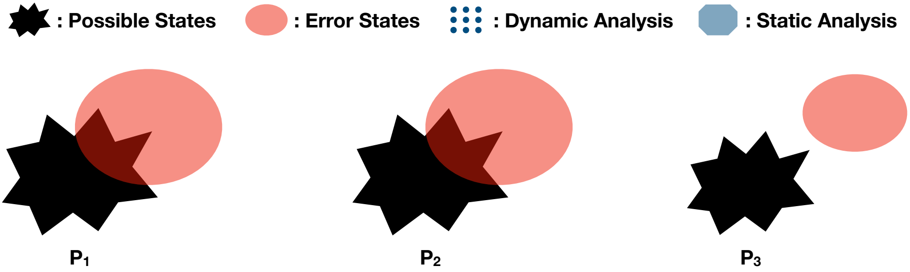

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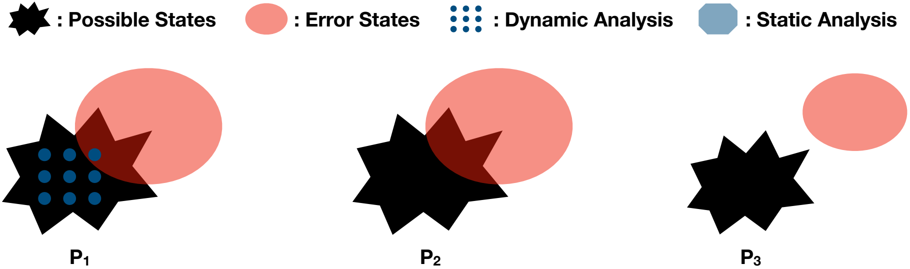

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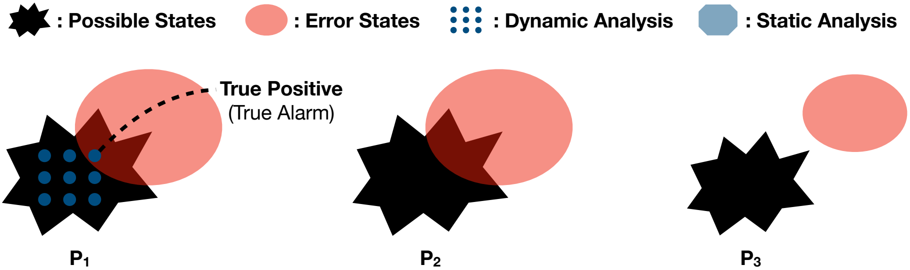

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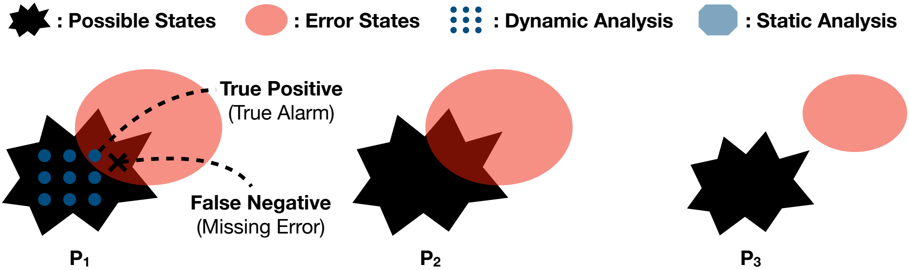

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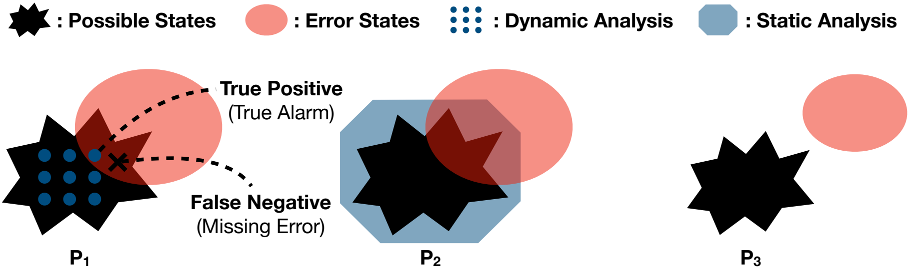

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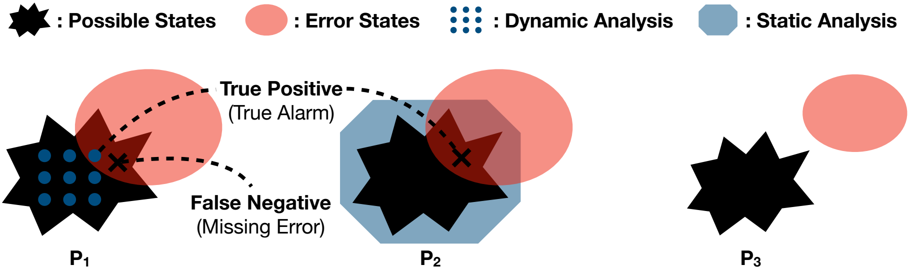

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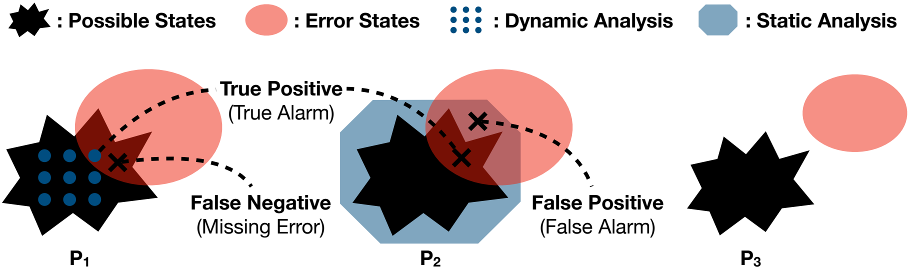

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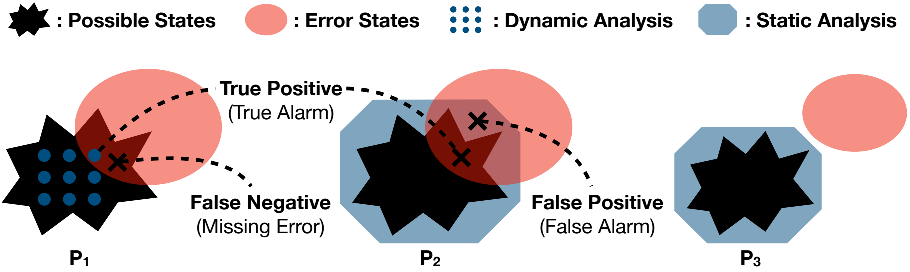

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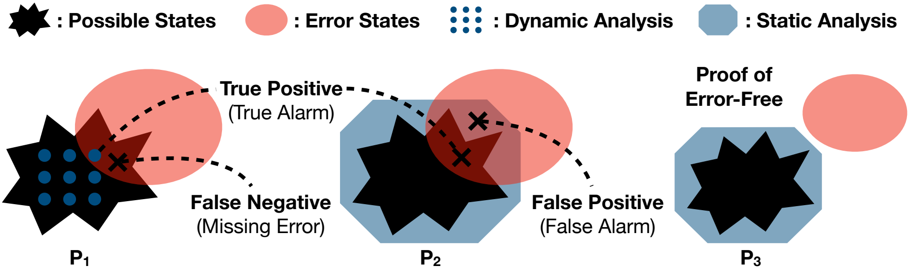

Contents

Types

Definition (Types)

A **type** is a set of values.

For example, the `Int`, `Boolean`, and `Int => Int` types are defined as the following sets of values in Scala.

$$\begin{array}{lcl} \texttt{Int} & = & \{n \in \mathbb{Z} \mid -2^{31} \leq n < 2^{31}\} \\ \texttt{Boolean} & = & \{\texttt{true}, \texttt{false}\} \\ \texttt{Int => Int} & = & \{f \mid f \text{ is a function from } \texttt{Int} \text{ to } \texttt{Int}\} \end{array}$$

```scala
val n: Int = 42               // 42   : Int
n + 1                         // 43   : Int
val b: Boolean = n > 10       // true : Boolean
def f(x: Int): Int = x + 1    // f    : Int => Int
f(42)                         // 43   : Int
```

Type Errors

Definition (Type Errors)

A **type error** occurs when a program tries to use a value having a type that is **incompatible** with the expected type.

For example, the following Scala program has type errors:

```scala
42 + true              // `Int` expected for `+`, but `Boolean` found
if (1) 2 else 3        // `Boolean` expected for `if`, but `Int` found
def f(x: Int): Int = x + 1
f(false)               // `Int` expected for `f`, but `Boolean` found
```

However, not all **run-time errors** are **type errors**:

```scala
42 / 0                 // `ArithmeticException` at run-time
case class A(k: Int)
val x: A = null
x.k                    // `NullPointerException` at run-time
```

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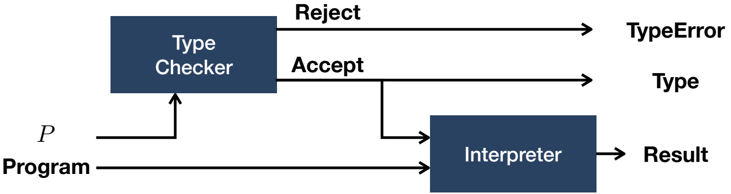
Definition (Type Checking)

Type checking is a kind of static analysis checking whether a given expression e is **well-typed**. A **type checker** returns the **type** of e if it is well-typed, or rejects it and reports the detected **type error** otherwise.



Type Soundness

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There are two categories of languages in the context of type system:

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Type systems in most statically-typed languages are designed to be **sound**.

Summary

1. Motivation: Safe Language Systems
    - Detecting Run-Time Errors
    - Dynamic vs Static Analysis
    - Soundness vs Completeness

2. Type Systems
    - Types
    - Type Errors
    - Type Checking
    - Type Soundness

Next Lecture

- Typed Languages

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