# Lecture 21 – Algebraic Data Types (1)

## COSE212: Programming Languages

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PLRG

2025 Fall

# Recall

- **TFAE** – FAE with **type system**.
  - **Type Checker** and **Typing Rules**
  - Interpreter and Natural Semantics

- **TRFAE** – RFAE with **type system**.
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- Let's learn **algebraic data types (ADTs)** and **pattern matching**!

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- **ATFAE** – TRFAE with **ADTs** and **pattern matching**.
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- **ATFAE** – TRFAE with **ADTs** and **pattern matching**.
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- In this lecture, we will focus on **Interpreter** and **Natural Semantics**.

# Contents

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# Recall: Types

**Definition (Types)**

A **type** is a set of values.

For example, the `Int`, `Boolean`, and `Int => Int` types are defined as the following sets of values in Scala.

$$\begin{array}{lcl} \texttt{Int} & = & \{n \in \mathbb{Z} \mid -2^{31} \leq n < 2^{31}\} \\ \texttt{Boolean} & = & \{\texttt{true}, \texttt{false}\} \\ \texttt{Int => Int} & = & \{f \mid f \text{ is a function from } \texttt{Int} \text{ to } \texttt{Int}\} \end{array}$$

```
val n: Int = 42               // 42   : Int
val b: Boolean = n > 10       // true : Boolean
def f(x: Int): Int = x + 1    // f    : Int => Int
f(42)                         // 43   : Int
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Is it possible to define a **new type** by **combining** existing types? **Yes!**

Product Types, Union Types, Sum Types, and Algebraic Data Types!

# Product Types

## Definition (Product Types)

A **product type** $(\tau_1, \ldots, \tau_n)$ is a set of values of the form $(v_1, \ldots, v_n)$ where $\tau_i$ is the type of $v_i$ for $1 \leq i \leq n$.

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For example, we can define product types in Scala as follows:

```scala
// A product type consisting of three different types
val triple: (Int, Boolean, String) = (42, true, "abc")

// A rectangle type with its width and height
type Rectangle = (Int, Int)
val rectangle: Rectangle = (10, 20)
val (w, h) = rectangle
val perimeter: Int = 2 * (w + h)      // 2 * (10 + 20) = 60
```

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For example, we can define union types in Scala as follows:

```scala
val a: Int | Boolean | String = 42
val b: Int | Boolean | String = true
val c: Int | Boolean | String = "abc"

type Square = Int                   // A square type
type Triangle = Int                 // A equilateral triangle type
val x: Square | Traingle = 42       // Is this a square or a triangle?
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How can we **discriminate** between a square and a triangle? **Sum types**!

# Sum Types

## Definition (Sum Types)

A **sum type** $x_1(\tau_1) + \ldots + x_n(\tau_n)$ consists of **variants** $x_i(\tau_i)$ for $1 \leq i \leq n$. For each variant $x_i(\tau_i)$, $x_i$ is the **constructor**, a function that takes a value $v$ of type $\tau_i$ and generates a value $x_i(v)$ of the sum type.

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It is corresponds to a **tagged union** of sets:

$$x_1(\tau_1) + \ldots + x_n(\tau_n) = \{x_i(v) \mid \exists 1 \leq i \leq n. \text{ s.t. } v \in \tau_i\}$$

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For example, we can define **sum types** in Scala as follows:

```scala
case class Square(side: Int)
case class Triangle(side: Int)
type Shape = Square | Triangle
val x: Shape = Square(42)     // It is a square
val y: Shape = Triangle(42)   // It is a triangle
```

Now, we can **discriminate** between a square and a triangle!

# Sum Types

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A **sum type** $x_1(\tau_1) + \ldots + x_n(\tau_n)$ consists of **variants** $x_i(\tau_i)$ for $1 \leq i \leq n$. For each variant $x_i(\tau_i)$, $x_i$ is the **constructor**, a function that takes a value $v$ of type $\tau_i$ and generates a value $x_i(v)$ of the sum type.

```scala
case class Square(side: Int)              // A variant for squares
case class Triangle(side: Int)            // A variant for triangles
type Shape = Square | Triangle
val x: Square | Triangle = Square(42)     // It is a square
val y: Square | Triangle = Triangle(42)   // It is a triangle

// `Square` is a constructor that takes an `Int` and generates a `Shape`
Square: Int => Shape

// `Square(42)` is a `Shape` value generated by `Square` constructor
Square(42): Shape
```

# Algebraic Data Types (ADTs)

**Definition (Algebraic Data Types (ADTs))**

An **algebraic data type** $x_1(\tau_{1,1}, \ldots, \tau_{1,m_1}) + \ldots + x_n(\tau_{n,1}, \ldots, \tau_{n,m_n})$ is a **recursive sum type** of **product types**.

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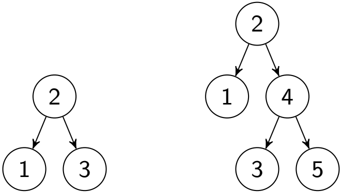
For example, we can define **algebraic data type** for trees in Scala:

```scala
enum Tree:
  case Leaf(v: Int)
  case Node(l: Tree, v: Int, r: Tree)

val t1: Tree = Node(Leaf(1), 2, Leaf(3))
val t2: Tree = Node(Leaf(1), 2, Node(Leaf(3), 4, Leaf(5)))
```



# Pattern Matching

## Definition (Pattern matching)

We can use **pattern matching** for algebraic data types to identify which variant of the sum type a value belongs to and extract the data it contains.

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For example, we can define a function `sum` that sums all the values in a tree using pattern matching (`match`) on the `Tree` type in Scala:

```
enum Tree:
  case Leaf(v: Int)
  case Node(l: Tree, v: Int, r: Tree)

def sum(t: Tree): Int = t match
  case Leaf(v)       => v
  case Node(l, v, r) => sum(l) + v + sum(r)

sum(Node(Leaf(1), 2, Leaf(3)))                        // 6
sum(Node(Leaf(1), 2, Node(Leaf(3), 4, Leaf(5))))      // 15
```

# Algebraic Data Types

Many functional languages support **algebraic data types**:

- Scala

```
enum Tree { case Leaf(v:Int); case Node(l:Tree, v:Int, r:Tree) }
```

- Haskell

```
data Tree = Leaf Int | Node Tree Int Tree
```

- Rust

```
enum Tree { Leaf(i32), Node(Tree, i32, Tree) }
```

- OCaml

```
type tree = Leaf of int | Node of tree * int * tree
```

- ...

# Contents

# ATFAE – TRFAE with ADTs and Pattern Matching

Now, let's extend TRFAE into ATFAE to support **algebraic data types** and **pattern matching**. (Assume that TRFAE supports multiple arguments for functions.)

```
/* ATFAE */
enum Tree {
  case Leaf(Number)
  case Node(Tree, Number, Tree)
}
Leaf(42) match {
  case Leaf(v)       => v
  case Node(l, v, r) => v
}
```

For ATFAE, we need to extend **expressions** of TRFAE with

1. **algebraic data types (ADTs)**
2. **pattern matching**
3. **type names**

# Concrete Syntax

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3. **type names**

We can extend the **concrete syntax** of TRFAE as follows:

```
// expressions
<expr> ::= ...
         | "enum" <id> "{" [ <variant> ";"? ]+ "}" ";"? <expr>
         | <expr> "match" "{" [ <mcase> ";"? ]+ "}"
// variants
<variant> ::= "case" <id> "(" ")"
            | "case" <id> "(" <type> [ "," <type> ]* ")"
// match cases
<mcase> ::= "case" <id> "(" ")" "=>" <expr>
          | "case" <id> "(" <id> [ "," <id> ]* ")" "=>" <expr>
// types
<type> ::= ... | <id>       // type names
```

# Abstract Syntax

$$
\begin{array}{llll}
\text{Expressions} & \mathbb{E} \ni e ::= \ldots & & \\
& \mid \texttt{enum}\ t\ \{\ [\texttt{case}\ x(\tau^*)]^+\ \};\ e & & (\texttt{TypeDef}) \\
& \mid e\ \texttt{match}\ \{\ [\texttt{case}\ x(x^*)\ \texttt{=>}\ e]^+\ \} & & (\texttt{Match}) \\
\text{Types} & \mathbb{T} \ni \tau ::= \ldots & & \\
& \mid t & & (\texttt{NameT}) \\
\text{Type Names} & t \in \mathbb{X}_t & & (\texttt{String})
\end{array}
$$

```
enum Expr:
  ...
  case TypeDef(name: String, varts: List[Variant], body: Expr)
  case Match(expr: Expr, mcases: List[MatchCase])

case class Variant(name: String, ptys: List[Type]):
case class MatchCase(name: String, params: List[String], body: Expr):

enum Type:
  ...
  case NameT(name: String)
```

# Abstract Syntax

```
/* ATFAE */
enum Tree {
  case Leaf(Number)
  case Node(Tree, Number, Tree)
}
Leaf(42) match {
  case Leaf(v)       => v
  case Node(l, v, r) => v
}
```

will be parsed to the following abstract syntax tree (AST) in Scala:

```
TypeDef("Tree", List(
    Variant("Leaf", List(NumT)),
    Variant("Node", List(NameT("Tree"), NumT, NameT("Tree")))
  ),
  Match(App(Id("Leaf"), List(Num(42))), List(
    MatchCase("Leaf", List("v"), Id("v")),
    MatchCase("Node", List("l", "v", "r"), Id("v")))))
```

# Contents

# Interpreter and Natural Semantics for ATFAE

For ATFAE, we need to 1) implement the **interpreter** with environments:

```
def interp(expr: Expr, env: Env): Value = ???
```

and 2) define the **natural semantics** with environments:

$$\boxed{\sigma \vdash e \Rightarrow v}$$

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def interp(expr: Expr, env: Env): Value = ???
```

and 2) define the **natural semantics** with environments:

$$\boxed{\sigma \vdash e \Rightarrow v}$$

with a new kind of values called **constructor values** and **variant values**:

$$
\begin{array}{llll}
\text{Values} & \mathbb{V} \ni v ::= n & (\texttt{NumV}) & \mid \langle x \rangle \quad (\texttt{ConstrV}) \\
& \quad\quad \mid b & (\texttt{BoolV}) & \mid x(v^*) \quad (\texttt{VariantV}) \\
& \quad\quad \mid \langle \lambda x.(e, \ldots, e), \sigma \rangle & (\texttt{CloV}) &
\end{array}
$$

```
enum Value:
  ...
  case ConstrV(name: String)
  case VariantV(name: String, values: List[Value])
```

# Algebraic Data Types

```scala
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case TypeDef(_, ws, body) =>
    interp(body, env ++ ws.map(w => w.name -> ConstrV(w.name)))
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\texttt{TypeDef}\ \frac{\sigma[x_1 \mapsto \langle x_1 \rangle, \ldots, x_n \mapsto \langle x_n \rangle] \vdash e \Rightarrow v}{\sigma \vdash \texttt{enum}\ t \left\{ \begin{array}{l} \texttt{case}\ x_1(\tau_{1,1}, \ldots, \tau_{1,m_1}) \\ \ldots \\ \texttt{case}\ x_n(\tau_{n,1}, \ldots, \tau_{n,m_n}) \end{array} \right\};\ e \Rightarrow v}$$

```
/* ATFAE */
enum Tree { case Leaf(Number); case Node(Tree, Number, Tree) }
Leaf(42) match { case Leaf(v) => v; case Node(l, v, r) => v }
```

# Algebraic Data Types

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case App(f, es) => interp(f, env) match
    case CloV(ps, b, fenv) => ...
    case ConstrV(name) => VariantV(name, es.map(interp(_, env)))
    case v             => error(s"not a function: ${v.str}")
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\texttt{App}_{\langle - \rangle}\ \frac{\sigma \vdash e_0 \Rightarrow \langle x \rangle \qquad \sigma \vdash e_1 \Rightarrow v_1 \qquad \ldots \qquad \sigma \vdash e_n \Rightarrow v_n}{\sigma \vdash e_0(e_1, \ldots, e_n) \Rightarrow x(v_1, \ldots, v_n)}$$

```
/* ATFAE */
enum Tree { case Leaf(Number); case Node(Tree, Number, Tree) }
Leaf(42) match { case Leaf(v) => v; case Node(l, v, r) => v }
```

# Pattern Matching

```scala
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Match(expr, cases) => interp(expr, env) match
    case VariantV(wname, vs) => cases.find(_.name == wname) match
      case Some(MatchCase(_, ps, b)) =>
        if (ps.length != vs.length) error("arity mismatch")
        interp(b, env ++ (ps zip vs))
      case None => error(s"no such case: $wname")
    case v => error(s"not a variant: ${v.str}")
```

$$\boxed{\sigma \vdash e \Rightarrow v}$$

$$\texttt{Match}\ \frac{1 \leq i \leq n \qquad \sigma \vdash e \Rightarrow x_i(v_1, \ldots, v_{m_i}) \qquad \forall j < i.\ x_j \neq x_i \qquad \sigma[x_{i,1} \mapsto v_1, \ldots, x_{i,m_i} \mapsto v_{m_i}] \vdash e_i \Rightarrow v}{\sigma \vdash e\ \texttt{match}\ \left\{\begin{array}{l} \texttt{case}\ x_1(x_{1,1}, \ldots, x_{1,m_1})\ \texttt{=>}\ e_1 \\ \ldots \\ \texttt{case}\ x_n(x_{n,1}, \ldots, x_{n,m_n})\ \texttt{=>}\ e_n \end{array}\right\} \Rightarrow v}$$

# Pattern Matching

There exists an **order** between the match cases: **first match wins!**

$$\texttt{Match}\ \frac{1 \leq i \leq n \qquad \sigma \vdash e \Rightarrow x_i(v_1, \ldots, v_{m_i}) \qquad \forall j < i.\ x_j \neq x_i \qquad \sigma[x_{i,1} \mapsto v_1, \ldots, x_{i,m_i} \mapsto v_{m_i}] \vdash e_i \Rightarrow v}{\sigma \vdash e\ \texttt{match} \left\{ \begin{array}{l} \texttt{case}\ x_1(x_{1,1}, \ldots, x_{1,m_1})\ \texttt{=>}\ e_1 \\ \ldots \\ \texttt{case}\ x_n(x_{n,1}, \ldots, x_{n,m_n})\ \texttt{=>}\ e_n \end{array} \right\} \Rightarrow v}$$

```
/* ATFAE */
enum Tree {
  case Leaf(Number)
  case Node(Tree, Number, Tree)
}
def f(t: Tree): Number = t match {
  case Leaf(v)       => v
  case Leaf(v)       => v + 1        // ignored
  case Node(l, v, r) => v
  case Node(l, v, r) => v + 1        // ignored
}; ...
```

# Example 1

```
/* ATFAE */
enum A { case B(Boolean); case C(Number) }
C(42) match { case B(b) => b; case C(n) => n < 0 }
```

$$
\texttt{TypeDef}\;\dfrac{\texttt{Match}\;\dfrac{\texttt{App}_{\langle - \rangle}\;\dfrac{\texttt{Id}\;\dfrac{C \in \mathsf{Domain}(\sigma_1)}{\sigma_1 \vdash C \Rightarrow \langle C \rangle} \quad \texttt{Num}\;\dfrac{}{\sigma_1 \vdash 42 \Rightarrow 42}}{\sigma_1 \vdash C(42) \Rightarrow C(42)} \qquad \texttt{Lt}\;\dfrac{\texttt{Id}\;\dfrac{n \in \mathsf{Domain}(\sigma_2)}{\sigma_2 \vdash n \Rightarrow 42} \quad \texttt{Num}\;\dfrac{}{\sigma_2 \vdash 0 \Rightarrow 0}}{\sigma_2 \vdash n \;\texttt{<}\; 0 \Rightarrow \texttt{false}}}{\sigma_1 \vdash C(42)\;\texttt{match}\;\left\{\begin{array}{l}\texttt{case}\;B(b)\;\texttt{=>}\;b\\ \texttt{case}\;C(n)\;\texttt{=>}\;n\;\texttt{<}\;0\end{array}\right\} \Rightarrow \texttt{false}}}{\varnothing \vdash \texttt{enum}\;A\;\left\{\begin{array}{l}\texttt{case}\;B(\texttt{bool})\\ \texttt{case}\;C(\texttt{num})\end{array}\right\};\;C(42)\;\texttt{match}\;\left\{\begin{array}{l}\texttt{case}\;B(b)\;\texttt{=>}\;b\\ \texttt{case}\;C(n)\;\texttt{=>}\;n\;\texttt{<}\;0\end{array}\right\} \Rightarrow \texttt{false}}
$$

where

$$
\begin{array}{lcl}
\sigma_1 & = & [B \mapsto \langle B \rangle, C \mapsto \langle C \rangle] \\
\sigma_2 & = & \sigma_1[n \mapsto 42]
\end{array}
$$

# Example 2

In **TFAE**, we cannot define `mkRec` because of the lack of **recursive types** in the language:

```
/* TFAE */

val mkRec = (body: (Number => Number) => Number => Number) => {
  val fX = (fY: ???) => {
    val f = (x: Number) => fY(fY)(x);
    body(f)
  };
  fX(fX)
};
val sum = mkRec((sum: Number => Number) => (n: Number) =>
  if (n < 1) 0
  else n + sum(n + -1));
sum(10)
```

# Example 2

Now, we can define `mkRec` in **ATFAE** because **algebraic data types** are **recursive types**:

```
/* ATFAE */
enum T { case T(T => Number => Number) }
val mkRec = (body: (Number => Number) => Number => Number) => {
  val fX = (fY: T) => {
    val f = (x: Number) => fY match { case T(fZ) => fZ(fY)(x) };
    body(f)
  };
  fX(T(fX))
};
val sum = mkRec((sum: Number => Number) => (n: Number) =>
  if (n < 1) 0
  else n + sum(n + -1));
sum(10)
```

# Example 3

We can define abstract syntax of AE using ADTs in ATFAE:

```
/* ATFAE */
enum Expr:
  case Num(number: Number)
  case Add(left: Expr, right: Expr)
  case Mul(left: Expr, right: Expr)
Add(Num(1), Mul(Num(2), Num(3)))        // 1 + 2 * 3
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We can define list type as well using ADTs in ATFAE:

```
/* ATFAE */
enum NumList:
  case Nil
  case Cons(head: Number, tail: NumList)
Cons(1, Cons(2, Cons(3, Nil)))          // list of 1, 2, and 3
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However, it only works for **monomorphic** lists (i.e., lists of numbers)

We will learn **parametric polymorphism** later in this course.

# Summary

1. Algebraic Data Types (ADTs) and Pattern Matching
    - Recall: Types
    - Product Types
    - Union Types
    - Sum Types
    - Algebraic Data Types (ADTs)
    - Pattern Matching
2. ATFAE – TRFAE with ADTs and Pattern Matching
    - Concrete Syntax
    - Abstract Syntax
3. Interpreter and Natural Semantics for ATFAE
    - Algebraic Data Types
    - Function Application
    - Pattern Matching
    - Examples

# Next Lecture

- Algebraic Data Types (2)

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