

Lecture 16 – First-Class Continuations

COSE212: Programming Languages

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- We will learn about **continuations** with the following topics:
 - **Continuations** (Lecture 14 & 15)
 - **First-Class Continuations** (Lecture 16)
 - **Compiling with continuations** (Lecture 17)
- A **continuation** represents the **rest of the computation**.
 - Continuation Passing Style (CPS)
 - Interpreter of FAE in CPS
 - Small-step operational semantics of FAE

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- KFAE – FAE with **first-class continuations**
 - Interpreter and Small-Step semantics

1. First-Class Continuations

2. KFAE – FAE with First-Class Continuations

Concrete/Abstract Syntax

3. Interpreter and Small-Step Semantics for KFAE

Recall: Interpreter and Small-Step Semantics for FAE

Interpreter and Small-Step Semantics for KFAE

First-Class Continuations

Function Application

Example 1

Example 2

4. Control Statements

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Recall: First-Class Citizen

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- ① **assigned** to a **variable**,
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- ① **assigned** to a **variable**,
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- ③ **returned** from a function.

For example, Scala supports **first-class functions**.

```
def inc(n: Int): Int = n + 1
// 1. We can assign a function to a variable.
val f: Int => Int = inc
// 2. We can pass a function as an argument to a function.
List(1, 2, 3).map(inc)           // List(2, 3, 4)
// 3. We can return a function from a function.
def addN(n: Int): Int => Int = m => n + m
val add3: Int => Int = addN(3)
add3(5)                         // 3 + 5 = 8
```

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For example, let's change the control flow of the following program:

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; Racket  
(* 2 (+ 3 5))
```

(Note that Racket uses the prefix notation (e.g., `(+ 1 2)`) instead of the infix notation (e.g., `1 + 2`).)

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```
; Racket
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```

Let's compare the evaluation of the two expressions.

The original expression is evaluated in the following order:

```
; Racket  
(* 2 (+ 3 5))
```

- ① Evaluate 2. (Result: 2)
- ② Evaluate 3. (Result: 3)
- ③ Evaluate 5. (Result: 5)
- ④ Add the results of step ② and ③. (Result: $3 + 5 = 8$)
- ⑤ Multiply the results of step ① and ④. (Result: $2 * 8 = 16$)

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What is the continuation of the expression `(+ 3 5)`? Step ⑤.

What is the continuation of the expression `5`?

The original expression is evaluated in the following order:

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What is the continuation of the expression `(+ 3 5)`? Step ⑤.

What is the continuation of the expression `5`? Steps ④ and ⑤.

Let's 1) **capture** the continuation of `(+ 3 5)` (i.e., ⑤) using `let/cc` and 2) **change** the **control flow** after evaluating 5 by using it as the continuation of 5 instead of the original one (i.e., ④ and ⑤).

We can change the program's control flow as follows:

```
; Racket  
(* 2 (let/cc k (+ 3 (k 5)))) ; first-class continuation with `let/cc`
```

- ① Evaluate 2. (Result: 2)
- ② Let k be the continuation of ② – ⑦. (k is Step ⑧)
- ③ Evaluate 3. (Result: 3)
- ④ Evaluate k . (Result: Step ⑧)
- ⑤ Evaluate 5. (Result: 5)
- ⑥ Call the result of step ④ with that of ⑤. (**Replace Cont.**)
- ⑦ Add the results of step ③ and ④ – ⑥. (Unreachable)
- ⑧ Multiply the results of step ① and ② – ⑦. (Result: $2 * 5 = 10$)

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- ⑧ Multiply the results of step ① and ② – ⑦. (Result: $2 * 5 = 10$)

It means that

- Step ② defines the continuation of steps ② – ⑦ as a value in k .
- Step ⑥ replaces the continuation of step ⑤ with k .

Some functional languages support **first-class continuations**.

- Racket

```
(* 2 (let/cc k (+ 3 (k 5))))           ; 2 * 5 = 10
```

- Ruby

```
2 * (callcc { |k| 3 + k.call(5) })    # 2 * 5 = 10
```

- Haskell

```
do
  x <- callCC $ \k -> do
    y <- k 5
    return $ 3 + y
  return $ 2 * x                       -- 2 * 5 = 10
```

- ...

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Now, let's extend FAE into KFAE with a new keyword `vcc` to capture the **first-class continuations**.

```
/* KFAE */  
2 * { vcc k; 3 + k(5) }
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```

Here is another example of KFAE:

```
/* KFAE */  
{  
  vcc done;  
  val f = {  
    vcc exit;  
    2 * done(1 + {  
      vcc k;  
      exit(k)  
    })  
  };  
  f(3) * 5  
}
```

For KFAE, we need to extend **expressions** of FAE with

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We can extend the **concrete syntax** of FAE as follows:

```
// expressions  
<expr> ::= ... | "vcc" <id> ";" <expr>
```

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① first-class continuations (`vcc`)

We can extend the **concrete syntax** of FAE as follows:

```
// expressions  
<expr> ::= ... | "vcc" <id> ";" <expr>
```

and the **abstract syntax** of FAE as follows:

Expressions $\mathbb{E} \ni e ::= \dots \mid \text{vcc } x; e \quad (\text{Vcc})$

```
enum Expr:  
  ...  
  // first-class continuations  
  case Vcc(name: String, body: Expr)
```

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Then, what is the expected result of the following KFAE expressions?

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    2 * done(1 + {  
      vcc k;         // k      = z => val f = { 2 * done(1 + z) }; f(3) * 5  
      exit(k)  
    })  
  };  
  f(3) * 5  
}                      // 4 (= 1 + 3)
```

In the previous lecture, we have defined the **first-order representation** of **continuations** with **value stack**:

```
enum Cont:
  case EmptyK
  case EvalK(env: Env, expr: Expr, k: Cont)
  case AddK(k: Cont)
  case MulK(k: Cont)
  case AppK(k: Cont)

type Stack = List[Value]
```

Continuations	$\mathbb{K} \ni \kappa ::= \square$	(EmptyK)
	$ (\sigma \vdash e) :: \kappa$	(EvalK)
	$ (+) :: \kappa$	(AddK)
	$ (*) :: \kappa$	(MulK)
	$ (@) :: \kappa$	(AppK)
Value Stacks	$\mathbb{S} \ni s ::= \blacksquare \mid v :: s$	(List[Value])

Then, we have defined the **reduction relation** $\rightarrow \in (\mathbb{K} \times \mathbb{S}) \times (\mathbb{K} \times \mathbb{S})$ between **states** consisting of pairs of **continuations** and **value stacks**:

```
def step(k: Cont, s: Stack): (Cont, Stack) = ???
```

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa' \parallel s' \rangle$$

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```

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa' \parallel s' \rangle$$

And the eval function **iteratively steps** the state until it reaches the empty continuation $\square$ and returns the single value in the value stack:

```
def eval(str: String): String =
  import Cont.*
  def aux(k: Cont, s: Stack): Value = step(k, s) match
    case (EmptyK, List(v)) => v
    case (k, s) => aux(k, s)
  aux(EvalK(Map.empty, Expr(str), EmptyK), List.empty).str
```

$$\langle (\emptyset \vdash e) :: \square \parallel \blacksquare \rangle \rightarrow^* \langle \square \parallel v :: \blacksquare \rangle$$

Now, let's extend the interpreter and small-step semantics for FAE to KFAE by adding the **first-class continuations**.

First, we need to extend the values of FAE with **continuation values** consisting of pairs of continuations and value stacks:

```
// values
enum Value:
  case NumV(number: BigInt)
  case CloV(param: String, body: Expr, env: Env)
  case ContV(cont: Cont, stack: Stack)
```

$$\begin{array}{ll} \text{Values } \mathbb{V} \ni v ::= n & (\text{NumV}) \\ & | \langle \lambda x.e, \sigma \rangle \quad (\text{CloV}) \\ & | \langle \kappa \parallel s \rangle \quad (\text{ContV}) \end{array}$$

Then, let's fill out the missing cases in the step function and rules for $\rightarrow$ in the small-step semantics of KFAE.

```
def step(k: Cont, s: Stack): (Cont, Stack) = (k, s) match
  case (EvalK(env, expr, k), s) => expr match
    ...
    case Vcc(x, b) => (EvalK(env + (x -> ContV(k, s)), b, k), s)
```

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa \parallel s \rangle$$

$$\text{Vcc} \quad \langle (\sigma \vdash \text{vcc } x; e) :: \kappa \parallel s \rangle \rightarrow \langle (\sigma[x \mapsto \langle \kappa \parallel s \rangle] \vdash e) :: \kappa \parallel s \rangle$$

It defines a new immutable binding x in the environment σ that maps to a **continuation value** $\langle \kappa \parallel s \rangle$, and then evaluates the body expression e in the extended environment $\sigma[x \mapsto \langle \kappa \parallel s \rangle]$.

```
def step(k: Cont, s: Stack): (Cont, Stack) = (k, s) match
  case (EvalK(env, expr, k), s) => expr match
    ...
    case App(f, e) => (EvalK(env, f, EvalK(env, e, AppK(k))), s)
  ...
  case (AppK(k), a :: f :: s) => f match
    case CloV(p, b, fenv) => (EvalK(fenv + (p -> a), b, k), s)
    case ContV(k1, s1)    => (k1, a :: s1)
    case v                 => error(s"not a function: ${v.str}")
```

$$\langle \kappa \parallel s \rangle \rightarrow \langle \kappa \parallel s \rangle$$

$$\begin{array}{ll} \text{App}_1 & \langle (\sigma \vdash e_1(e_2)) :: \kappa \parallel s \rangle \rightarrow \langle (\sigma \vdash e_1) :: (\sigma \vdash e_2) :: (@ :: \kappa \parallel s) \rangle \\ \text{App}_{2,\lambda} & \langle (@ :: \kappa \parallel v_2 :: \langle \lambda x.e, \sigma \rangle :: s) \rangle \rightarrow \langle (\sigma[x \mapsto v_2] \vdash e) :: \kappa \parallel s \rangle \\ \text{App}_{2,\kappa} & \langle (@ :: \kappa \parallel v_2 :: \langle \kappa' \parallel s' \rangle :: s) \rangle \rightarrow \langle \kappa' \parallel v_2 :: s' \rangle \end{array}$$

The new $\text{App}_{2,\kappa}$ rule handles when the function expression evaluates to a continuation value $\langle \kappa' \parallel s' \rangle$. It changes the control flow to the continuation κ' with the given argument value v_2 and the value stack s' .

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$$\begin{array}{l} \text{(Mul}_1\text{)} \quad \langle (\emptyset \vdash 2 * (\text{vcc } x; (3 + x(5)))) \rangle :: \square \quad \parallel \blacksquare \quad \rangle \\ \rightarrow \quad \langle (\emptyset \vdash 2) \rangle :: \langle (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) \rangle :: (*) :: \square \quad \parallel \blacksquare \quad \rangle \end{array}$$

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$$\text{where } \begin{cases} \sigma_0 = [x \mapsto \langle \kappa_0 \mid s_0 \rangle] \\ \kappa_0 = (*) :: \square \\ s_0 = 2 :: \blacksquare \end{cases}$$

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(Mul_1) $\rightarrow$	$\langle (\emptyset \vdash 2) :: (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel$	■	$\rangle$
(Num) $\rightarrow$	$\langle (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel$	$2 ::$	■ $\rangle$
(Vcc) $\rightarrow$	$\langle (\sigma_0 \vdash (3 + x(5))) :: (*) :: \square \rangle$	$\parallel$	$2 ::$	■ $\rangle$
(Add_1) $\rightarrow$	$\langle (\sigma_0 \vdash 3) :: (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel$	$2 ::$	■ $\rangle$

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(Mul_1)	$\langle (\emptyset \vdash 2 * (\text{vcc } x; (3 + x(5)))) \rangle :: \square$	$\parallel$	■	$\rangle$
$\rightarrow$	$\langle (\emptyset \vdash 2) :: (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) \rangle :: (*) :: \square$	$\parallel$	■	$\rangle$
(Num)	$\langle (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) \rangle :: (*) :: \square$	$\parallel$	$2 ::$	■ $\rangle$
(Vcc)	$\langle (\sigma_0 \vdash (3 + x(5))) \rangle :: (*) :: \square$	$\parallel$	$2 ::$	■ $\rangle$
$\rightarrow$	$\langle (\sigma_0 \vdash 3) :: (\sigma_0 \vdash x(5)) \rangle :: (+) :: (*) :: \square$	$\parallel$	$2 ::$	■ $\rangle$
(Add_1)	$\langle (\sigma_0 \vdash x(5)) \rangle :: (+) :: (*) :: \square$	$\parallel$	$3 :: 2 ::$	■ $\rangle$
$\rightarrow$				

$$\text{where } \begin{cases} \sigma_0 = [x \mapsto \langle \kappa_0 \parallel s_0 \rangle] \\ \kappa_0 = (*) :: \square \\ s_0 = 2 :: \blacksquare \end{cases}$$

Example 1

Let's interpret the expression $2 * (\text{vcc } x; (3 + x(5)))$:

	$\langle (\emptyset \vdash 2 * (\text{vcc } x; (3 + x(5)))) :: \square \rangle$	$\parallel$	■	$\rangle$
(Mul ₁) →	$\langle (\emptyset \vdash 2) :: (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel$	■	$\rangle$
(Num) →	$\langle (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel$	2 :: ■	$\rangle$
(Vcc) →	$\langle (\sigma_0 \vdash (3 + x(5))) :: (*) :: \square \rangle$	$\parallel$	2 :: ■	$\rangle$
(Add ₁) →	$\langle (\sigma_0 \vdash 3) :: (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel$	2 :: ■	$\rangle$
(Num) →	$\langle (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel$	3 :: 2 :: ■	$\rangle$
(App ₁) →	$\langle (\sigma_0 \vdash x) :: (\sigma_0 \vdash 5) :: (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	3 :: 2 :: ■	$\rangle$

$$\text{where } \begin{cases} \sigma_0 = [x \mapsto \langle \kappa_0 \parallel s_0 \rangle] \\ \kappa_0 = (*) :: \square \\ s_0 = 2 :: \blacksquare \end{cases}$$

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Let's interpret the expression $2 * (\text{vcc } x; (3 + x(5)))$:

	$\langle (\emptyset \vdash 2 * (\text{vcc } x; (3 + x(5)))) :: \square \rangle$	$\parallel \blacksquare$	$\rangle$
(Mul_1) $\rightarrow$	$\langle (\emptyset \vdash 2) :: (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel \blacksquare$	$\rangle$
(Num) $\rightarrow$	$\langle (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel 2 :: \blacksquare$	$\rangle$
(Vcc) $\rightarrow$	$\langle (\sigma_0 \vdash (3 + x(5))) :: (*) :: \square \rangle$	$\parallel 2 :: \blacksquare$	$\rangle$
(Add_1) $\rightarrow$	$\langle (\sigma_0 \vdash 3) :: (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel 2 :: \blacksquare$	$\rangle$
(Num) $\rightarrow$	$\langle (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel 3 :: 2 :: \blacksquare$	$\rangle$
(App_1) $\rightarrow$	$\langle (\sigma_0 \vdash x) :: (\sigma_0 \vdash 5) :: (@) :: (+) :: (*) :: \square \rangle$	$\parallel 3 :: 2 :: \blacksquare$	$\rangle$
(Id) $\rightarrow$	$\langle (\sigma_0 \vdash 5) :: (@) :: (+) :: (*) :: \square \rangle$	$\parallel \langle \kappa_0 \parallel s_0 \rangle :: 3 :: 2 :: \blacksquare$	$\rangle$

$$\text{where } \begin{cases} \sigma_0 = [x \mapsto \langle \kappa_0 \parallel s_0 \rangle] \\ \kappa_0 = (*) :: \square \\ s_0 = 2 :: \blacksquare \end{cases}$$

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	$\langle (\emptyset \vdash 2 * (\text{vcc } x; (3 + x(5)))) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
(Mul_1) $\rightarrow$	$\langle (\emptyset \vdash 2) :: (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
(Num) $\rightarrow$	$\langle (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel$	$2 :: \blacksquare$	$\rangle$
(Vcc) $\rightarrow$	$\langle (\sigma_0 \vdash (3 + x(5))) :: (*) :: \square \rangle$	$\parallel$	$2 :: \blacksquare$	$\rangle$
(Add_1) $\rightarrow$	$\langle (\sigma_0 \vdash 3) :: (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel$	$2 :: \blacksquare$	$\rangle$
(Num) $\rightarrow$	$\langle (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel$	$3 :: 2 :: \blacksquare$	$\rangle$
(App_1) $\rightarrow$	$\langle (\sigma_0 \vdash x) :: (\sigma_0 \vdash 5) :: (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	$3 :: 2 :: \blacksquare$	$\rangle$
(Id) $\rightarrow$	$\langle (\sigma_0 \vdash 5) :: (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	$\langle \kappa_0 \parallel s_0 \rangle :: 3 :: 2 :: \blacksquare$	$\rangle$
(Num) $\rightarrow$	$\langle (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	$5 :: \langle \kappa_0 \parallel s_0 \rangle :: 3 :: 2 :: \blacksquare$	$\rangle$

$$\text{where } \begin{cases} \sigma_0 = [x \mapsto \langle \kappa_0 \parallel s_0 \rangle] \\ \kappa_0 = (*) :: \square \\ s_0 = 2 :: \blacksquare \end{cases}$$

Example 1

Let's interpret the expression $2 * (\text{vcc } x; (3 + x(5)))$:

(Mul_1)	$\langle (\emptyset \vdash 2 * (\text{vcc } x; (3 + x(5)))) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
$\rightarrow$	$\langle (\emptyset \vdash 2) :: (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
(Num)	$\langle (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel$	$2 :: \blacksquare$	$\rangle$
(Vcc)	$\langle (\sigma_0 \vdash (3 + x(5))) :: (*) :: \square \rangle$	$\parallel$	$2 :: \blacksquare$	$\rangle$
$\rightarrow$	$\langle (\sigma_0 \vdash 3) :: (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel$	$2 :: \blacksquare$	$\rangle$
(Add_1)	$\langle (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel$	$3 :: 2 :: \blacksquare$	$\rangle$
$\rightarrow$	$\langle (\sigma_0 \vdash x) :: (\sigma_0 \vdash 5) :: (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	$3 :: 2 :: \blacksquare$	$\rangle$
(App_1)	$\langle (\sigma_0 \vdash 5) :: (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	$\langle \kappa_0 \parallel s_0 \rangle :: 3 :: 2 :: \blacksquare$	$\rangle$
$\rightarrow$	$\langle (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	$5 :: \langle \kappa_0 \parallel s_0 \rangle :: 3 :: 2 :: \blacksquare$	$\rangle$
(Num)	$\langle (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	$5 :: 2 :: \blacksquare$	$\rangle$
$\rightarrow$	$\langle (*) :: \square \rangle$	$\parallel$	$5 :: 2 :: \blacksquare$	$\rangle$
$(\text{App}_{2,\kappa})$				

$$\text{where } \begin{cases} \sigma_0 = [x \mapsto \langle \kappa_0 \parallel s_0 \rangle] \\ \kappa_0 = (*) :: \square \\ s_0 = 2 :: \blacksquare \end{cases}$$

Example 1

Let's interpret the expression $2 * (\text{vcc } x; (3 + x(5)))$:

	$\langle (\emptyset \vdash 2 * (\text{vcc } x; (3 + x(5)))) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
(Mul_1)	$\rightarrow \langle (\emptyset \vdash 2) :: (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
(Num)	$\rightarrow \langle (\emptyset \vdash (\text{vcc } x; (3 + x(5)))) :: (*) :: \square \rangle$	$\parallel$	$2 :: \blacksquare$	$\rangle$
(Vcc)	$\rightarrow \langle (\sigma_0 \vdash (3 + x(5))) :: (*) :: \square \rangle$	$\parallel$	$2 :: \blacksquare$	$\rangle$
(Add_1)	$\rightarrow \langle (\sigma_0 \vdash 3) :: (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel$	$2 :: \blacksquare$	$\rangle$
(Num)	$\rightarrow \langle (\sigma_0 \vdash x(5)) :: (+) :: (*) :: \square \rangle$	$\parallel$	$3 :: 2 :: \blacksquare$	$\rangle$
(App_1)	$\rightarrow \langle (\sigma_0 \vdash x) :: (\sigma_0 \vdash 5) :: (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	$3 :: 2 :: \blacksquare$	$\rangle$
(Id)	$\rightarrow \langle (\sigma_0 \vdash 5) :: (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	$\langle \kappa_0 \parallel s_0 \rangle :: 3 :: 2 :: \blacksquare$	$\rangle$
(Num)	$\rightarrow \langle (@) :: (+) :: (*) :: \square \rangle$	$\parallel$	$5 :: \langle \kappa_0 \parallel s_0 \rangle :: 3 :: 2 :: \blacksquare$	$\rangle$
$(\text{App}_{2,\kappa})$	$\rightarrow \langle (*) :: \square \rangle$	$\parallel$	$5 :: 2 :: \blacksquare$	$\rangle$
(Mul_2)	$\rightarrow \langle \square \rangle$	$\parallel$	$10 :: \blacksquare$	$\rangle$

$$\text{where } \begin{cases} \sigma_0 = [\mathbf{x} \mapsto \langle \kappa_0 \parallel s_0 \rangle] \\ \kappa_0 = (*) :: \square \\ s_0 = 2 :: \blacksquare \end{cases}$$

Example 2

Let's interpret the expression $(\lambda x. (\text{vcc } r; r(x + 1) * 2))(3)$:

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Let's interpret the expression $(\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)$:

$$\langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)) :: \square \quad \quad \quad || \quad \blacksquare \quad \quad \quad \rangle$$

where {

Example 2

Let's interpret the expression $(\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)$:

$$\begin{array}{lcl} (\text{App}_1) & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)) :: \square & \parallel \blacksquare \rangle \\ \xrightarrow{\quad} & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))) :: (\emptyset \vdash 3) :: (@) :: \square & \parallel \blacksquare \rangle \end{array}$$

where {

Example 2

Let's interpret the expression $(\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)$:

$$\begin{array}{lcl}
 (\text{App}_1) & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)) :: \square & \parallel \blacksquare \rangle \\
 \rightarrow & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))) :: (\emptyset \vdash 3) :: (@) :: \square & \parallel \blacksquare \rangle \\
 (\text{Fun}) & \rightarrow & \langle (\emptyset \vdash 3) :: (@) :: \square \parallel \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle
 \end{array}$$

$$\text{where } \left\{ \begin{array}{l} e_0 = \text{vcc } r; r(x + 1) * 2 \end{array} \right.$$

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$$\begin{array}{ll}
 (\text{App}_1) & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)) :: \square \quad \parallel \blacksquare \rangle \\
 \xrightarrow{\quad} & \\
 (\text{Fun}) & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))) :: (\emptyset \vdash 3) :: (@) :: \square \parallel \blacksquare \rangle \\
 \xrightarrow{\quad} & \\
 (\text{Num}) & \langle (\emptyset \vdash 3) :: (@) :: \square \quad \parallel \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 \xrightarrow{\quad} & \\
 & \langle (@) :: \square \quad \parallel 3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle
 \end{array}$$

$$\text{where } \left\{ \begin{array}{l} e_0 = \text{vcc } r; r(x + 1) * 2 \end{array} \right.$$

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Let's interpret the expression $(\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)$:

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 (\text{App}_1) \rightarrow & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))) :: (\emptyset \vdash 3) :: (@) :: \square & || \blacksquare \rangle \\
 (\text{Fun}) \rightarrow & \langle (\emptyset \vdash 3) :: (@) :: \square & || \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 (\text{Num}) \rightarrow & \langle (@) :: \square & || 3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 (\text{App}_{2,\lambda}) \rightarrow & \langle (\sigma_0 \vdash \text{vcc } r; r(x + 1) * 2) :: \square & || \blacksquare \rangle
 \end{array}$$

$$\text{where } \begin{cases} e_0 &= \text{vcc } r; r(x + 1) * 2 \\ \sigma_0 &= [x \mapsto 3] \end{cases}$$

Example 2

Let's interpret the expression $(\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)$:

$$\begin{array}{llll}
 & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)) :: \square & || & \blacksquare \rangle \\
 (\text{App}_1) \rightarrow & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))) :: (\emptyset \vdash 3) :: (@) :: \square & || & \blacksquare \rangle \\
 (\text{Fun}) \rightarrow & \langle (\emptyset \vdash 3) :: (@) :: \square & || & \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 (\text{Num}) \rightarrow & \langle (@) :: \square & || & 3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 (\text{App}_{2,\lambda}) \rightarrow & \langle (\sigma_0 \vdash \text{vcc } r; r(x + 1) * 2) :: \square & || & \blacksquare \rangle \\
 (\text{Vcc}) \rightarrow & \langle (\sigma_1 \vdash r(x + 1) * 2) :: \square & || & \blacksquare \rangle
 \end{array}$$

$$\text{where } \begin{cases} e_0 &= \text{vcc } r; r(x + 1) * 2 \\ \sigma_0 &= [x \mapsto 3] \\ \sigma_1 &= \sigma_0[r \mapsto \langle \square || \blacksquare \rangle] \end{cases}$$

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 (\text{App}_1) \xrightarrow{\quad} & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))) :: (\emptyset \vdash 3) :: (@) :: \square & || \blacksquare \rangle \\
 (\text{Fun}) \xrightarrow{\quad} & \langle (\emptyset \vdash 3) :: (@) :: \square & || \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 (\text{Num}) \xrightarrow{\quad} & \langle (@) :: \square & || 3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 (\text{App}_{2,\lambda}) \xrightarrow{\quad} & \langle (\sigma_0 \vdash \text{vcc } r; r(x + 1) * 2) :: \square & || \blacksquare \rangle \\
 (\text{Vcc}) \xrightarrow{\quad} & \langle (\sigma_1 \vdash r(x + 1) * 2) :: \square & || \blacksquare \rangle \\
 (\text{Mul}_1) \xrightarrow{\quad} & \langle (\sigma_1 \vdash r(x + 1)) :: (\sigma_1 \vdash 2) :: (*) :: \square & || \blacksquare \rangle
 \end{array}$$

$$\text{where } \begin{cases} e_0 &= \text{vcc } r; r(x + 1) * 2 \\ \sigma_0 &= [x \mapsto 3] \\ \sigma_1 &= \sigma_0[r \mapsto \langle \square || \blacksquare \rangle] \end{cases}$$

Example 2

Let's interpret the expression $(\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)$:

(App_1)	$\langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
$\xrightarrow{}$	$\langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))) :: (\emptyset \vdash 3) :: (@) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
(Fun)	$\langle (\emptyset \vdash 3) :: (@) :: \square \rangle$	$\parallel$	$\langle \lambda x.e_0, \emptyset \rangle :: \blacksquare$	$\rangle$
$\xrightarrow{}$	$\langle (@) :: \square \rangle$	$\parallel$	$3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare$	$\rangle$
(Num)	$\langle (@) :: \square \rangle$	$\parallel$	$3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare$	$\rangle$
$\xrightarrow{}$	$\langle (@) :: \square \rangle$	$\parallel$	$3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare$	$\rangle$
$(\text{App}_{2,\lambda})$	$\langle (\sigma_0 \vdash \text{vcc } r; r(x + 1) * 2) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
$\xrightarrow{}$	$\langle (\sigma_1 \vdash r(x + 1) * 2) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
(Vcc)	$\langle (\sigma_1 \vdash r(x + 1) * 2) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
$\xrightarrow{}$	$\langle (\sigma_1 \vdash r(x + 1) * 2) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
(Mul_1)	$\langle (\sigma_1 \vdash r(x + 1)) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
$\xrightarrow{}$	$\langle (\sigma_1 \vdash r(x + 1)) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
(App_1)	$\langle (\sigma_1 \vdash r) :: (\sigma_1 \vdash x + 1) :: (@) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
$\xrightarrow{}$	$\langle (\sigma_1 \vdash r) :: (\sigma_1 \vdash x + 1) :: (@) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle$	$\parallel$	$\blacksquare$	$\rangle$
$\dots$				

$$\text{where } \begin{cases} e_0 &= \text{vcc } r; r(x + 1) * 2 \\ \sigma_0 &= [x \mapsto 3] \\ \sigma_1 &= \sigma_0[r \mapsto \langle \square \parallel \blacksquare \rangle] \end{cases}$$

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Let's interpret the expression $(\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)$:

$$\begin{array}{lll}
 (\text{App}_1) & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)) :: \square & || \blacksquare \rangle \\
 \rightarrow & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))) :: (\emptyset \vdash 3) :: (@) :: \square & || \blacksquare \rangle \\
 (\text{Fun}) & \rightarrow & \langle (\emptyset \vdash 3) :: (@) :: \square & || \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 (\text{Num}) & \rightarrow & \langle (@) :: \square & || 3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \rangle \\
 (\text{App}_{2,\lambda}) & \rightarrow & \langle (\sigma_0 \vdash \text{vcc } r; r(x + 1) * 2) :: \square & || \blacksquare \rangle \\
 (\text{Vcc}) & \rightarrow & \langle (\sigma_1 \vdash r(x + 1) * 2) :: \square & || \blacksquare \rangle \\
 (\text{Mul}_1) & \rightarrow & \langle (\sigma_1 \vdash r(x + 1)) :: (\sigma_1 \vdash 2) :: (*) :: \square & || \blacksquare \rangle \\
 (\text{App}_1) & \rightarrow & \langle (\sigma_1 \vdash r) :: (\sigma_1 \vdash x + 1) :: (@) :: (\sigma_1 \vdash 2) :: (*) :: \square & || \blacksquare \rangle \\
 \dots & & & \\
 \rightarrow^* & \langle (@) :: (\sigma_1 \vdash 2) :: (*) :: \square & || 4 :: \langle \square || \blacksquare \rangle :: \blacksquare \rangle
 \end{array}$$

$$\text{where } \begin{cases} e_0 & = & \text{vcc } r; r(x + 1) * 2 \\ \sigma_0 & = & [x \mapsto 3] \\ \sigma_1 & = & \sigma_0[r \mapsto \langle \square || \blacksquare \rangle] \end{cases}$$

Example 2

Let's interpret the expression $(\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)$:

$$\begin{array}{ll}
 (\text{App}_1) & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))(3)) :: \square \rangle \quad || \quad \blacksquare \quad \rangle \\
 \rightarrow & \langle (\emptyset \vdash (\lambda x.(\text{vcc } r; r(x + 1) * 2))) :: (\emptyset \vdash 3) :: (@) :: \square \rangle \quad || \quad \blacksquare \quad \rangle \\
 (\text{Fun}) & \langle (\emptyset \vdash 3) :: (@) :: \square \rangle \quad || \quad \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \quad \rangle \\
 \rightarrow & \langle (@) :: \square \rangle \quad || \quad 3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \quad \rangle \\
 (\text{Num}) & \langle (@) :: \square \rangle \quad || \quad 3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \quad \rangle \\
 \rightarrow & \langle (@) :: \square \rangle \quad || \quad 3 :: \langle \lambda x.e_0, \emptyset \rangle :: \blacksquare \quad \rangle \\
 (\text{App}_{2,\lambda}) & \langle (\sigma_0 \vdash \text{vcc } r; r(x + 1) * 2) :: \square \rangle \quad || \quad \blacksquare \quad \rangle \\
 \rightarrow & \langle (\sigma_1 \vdash r(x + 1) * 2) :: \square \rangle \quad || \quad \blacksquare \quad \rangle \\
 (\text{Vcc}) & \langle (\sigma_1 \vdash r(x + 1) * 2) :: \square \rangle \quad || \quad \blacksquare \quad \rangle \\
 \rightarrow & \langle (\sigma_1 \vdash r(x + 1)) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle \quad || \quad \blacksquare \quad \rangle \\
 (\text{Mul}_1) & \langle (\sigma_1 \vdash r(x + 1)) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle \quad || \quad \blacksquare \quad \rangle \\
 \rightarrow & \langle (\sigma_1 \vdash r) :: (\sigma_1 \vdash x + 1) :: (@) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle \quad || \quad \blacksquare \quad \rangle \\
 (\text{App}_1) & \langle (\sigma_1 \vdash r) :: (\sigma_1 \vdash x + 1) :: (@) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle \quad || \quad \blacksquare \quad \rangle \\
 \rightarrow & \dots \\
 \rightarrow^* & \langle (@) :: (\sigma_1 \vdash 2) :: (*) :: \square \rangle \quad || \quad 4 :: \langle \square || \blacksquare \rangle :: \blacksquare \quad \rangle \\
 (\text{App}_{2,\kappa}) & \langle \square \rangle \quad || \quad 4 :: \blacksquare \quad \rangle
 \end{array}$$

$$\text{where } \begin{cases} e_0 & = & \text{vcc } r; r(x + 1) * 2 \\ \sigma_0 & = & [x \mapsto 3] \\ \sigma_1 & = & \sigma_0[r \mapsto \langle \square || \blacksquare \rangle] \end{cases}$$

1. First-Class Continuations
2. KFAE – FAE with First-Class Continuations
Concrete/Abstract Syntax
3. Interpreter and Small-Step Semantics for KFAE
Recall: Interpreter and Small-Step Semantics for FAE
Interpreter and Small-Step Semantics for KFAE
First-Class Continuations
Function Application
Example 1
Example 2
4. Control Statements

Many real-world programming languages support **control statements** to change the **control-flow** of a program.

Many real-world programming languages support **control statements** to change the **control-flow** of a program.

For example, C++ supports **break**, **continue**, and **return** statements:

```
int sumEvenUntilZero(int xs[], int len) {  
    if (len <= 0) return 0;           // directly return 0 if len <= 0  
    int sum = 0;  
    for (int i = 0; i < len; i++) {  
        if (xs[i] == 0) break;        // stop the loop if xs[i] == 0  
        if (xs[i] % 2 == 1) continue; // skip the rest if xs[i] is odd  
        sum += xs[i];  
    }  
    return sum;                       // finally return the sum  
}  
  
int xs[] = {4, 1, 3, 2, 0, 6, 5, 8};  
sumEvenUntilZero(xs, 8);             // 4 + 2 = 6
```

Let's represent them using **first-class continuations**!

- `return` statement:

```
x => body
```


- **return** statement:

```
x => body
```

means

```
x => { vcc return;  
      body // return(e) directly returns e to the caller  
}
```

- **return** statement:

```
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means

```
x => { vcc return;  
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```

- **break** and **continue** statements:

```
while (cond) body
```

- **return** statement:

```
x => body
```

means

```
x => { vcc return;  
      body // return(e) directly returns e to the caller  
    }
```

- **break** and **continue** statements:

```
while (cond) body
```

means

```
{ vcc break;  
  while (cond) { vcc continue;  
    body // continue(e)/break(e) jumps to the next/end of the loop  
  }  
}
```

We can represent other control statements similarly, but think for yourself!

- exception in Python

```
try:
    x = y / z
except ZeroDivisionError:
    x = 0
```

- generator in JavaScript

```
const foo = function* () { yield 'a'; yield 'b'; yield 'c'; };
let str = '';
for (const c of foo()) { str = str + c; }
str // 'abc'
```

- coroutines in Kotlin
- async/await in C#
- ...

1. First-Class Continuations

2. KFAE – FAE with First-Class Continuations

Concrete/Abstract Syntax

3. Interpreter and Small-Step Semantics for KFAE

Recall: Interpreter and Small-Step Semantics for FAE

Interpreter and Small-Step Semantics for KFAE

First-Class Continuations

Function Application

Example 1

Example 2

4. Control Statements

<https://github.com/ku-plrg-classroom/docs/tree/main/cose212/kfae>

- Please see above document on GitHub:
 - Implement step function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise, so we **strongly recommend** solving it by yourself.

- Compiling with Continuations

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