Lecture 18 – Type Systems

COSE212: Programming Languages

Jihyeok Park

PLRG

2026 Fall

Recall

- We learned about **continuations** with the following topics:
  - **Continuations** (Lecture 14 & 15)
  - **First-Class Continuations** (Lecture 16)
  - **Compiling with continuations** (Lecture 17)

Recall

- We learned about **continuations** with the following topics:
  - **Continuations** (Lecture 14 & 15)
  - **First-Class Continuations** (Lecture 16)
  - **Compiling with continuations** (Lecture 17)

- From now on, we will learn about **type systems** with the following topics until the end of the semester:
  - Typed Languages
  - Typing Recursive Functions
  - Algebraic Data Types
  - Parametric Polymorphism
  - Subtype Polymorphism
  - Type Inference

Recall

- We learned about **continuations** with the following topics:
  - **Continuations** (Lecture 14 & 15)
  - **First-Class Continuations** (Lecture 16)
  - **Compiling with continuations** (Lecture 17)

- From now on, we will learn about **type systems** with the following topics until the end of the semester:
  - Typed Languages
  - Typing Recursive Functions
  - Algebraic Data Types
  - Parametric Polymorphism
  - Subtype Polymorphism
  - Type Inference

- In this lecture, we will focus on the **motivation** and **basic concepts** of type systems.

Contents

Contents

Run-Time Errors

So far, we have designed diverse programming languages with:

- **Syntax**: a grammar that defines the structure of programs
- **Semantics**: a set of rules that defines the meaning of programs

and implemented their **interpreters** in Scala:

Run-Time Errors

So far, we have designed diverse programming languages with:

- **Syntax**: a grammar that defines the structure of programs
- **Semantics**: a set of rules that defines the meaning of programs

and implemented their **interpreters** in Scala:

However, we don't have any automatic system to **check** whether a program is evaluated without any **run-time errors**.

Run-Time Errors

So far, we have designed diverse programming languages with:

- **Syntax**: a grammar that defines the structure of programs
- **Semantics**: a set of rules that defines the meaning of programs

and implemented their **interpreters** in Scala:

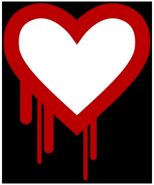
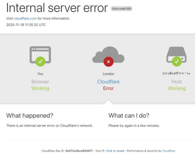
However, we don't have any automatic system to **check** whether a program is evaluated without any **run-time errors**.

For example, following FAE expressions are syntactically correct, but they throw **run-time errors**:

```
/* FAE */
x * 42                  // error: free identifier
0 + (x => x)            // error: cannot add a function
1(2)                    // error: cannot apply a number
```

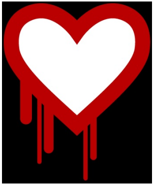
Errors in Critical Software

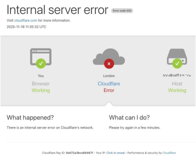
Unexpected **run-time errors** in critical software cause serious problems:

			
Rocket Ariane 5 (1996)	**Security** Heartbleed (2014)	**Global IT** CrowdStrike (2024)	**Cloud** Cloudflare (2025)

Errors in Critical Software

Unexpected **run-time errors** in critical software cause serious problems:



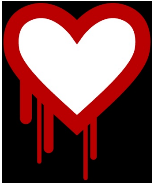
Rocket	**Security**	**Global IT**	**Cloud**
Ariane 5	Heartbleed	CrowdStrike	Cloudflare
(1996)	(2014)	(2024)	(2025)

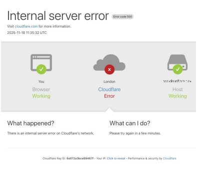
Then, how can we **prevent** such errors?

Errors in Critical Software

PLRG

Unexpected **run-time errors** in critical software cause serious problems:





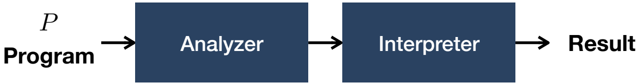
Rocket	**Security**	**Global IT**	**Cloud**
Ariane 5	Heartbleed	CrowdStrike	Cloudflare
(1996)	(2014)	(2024)	(2025)

Then, how can we **prevent** such errors?

Can we **automatically** check whether a program does not have any **run-time errors**?

Detecting Run-Time Errors

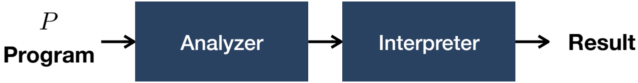
We can use various **analysis** techniques to detect run-time errors:



An **analyzer** is a program that takes a program as an input and determines whether the program has a certain property. In this case, the property is **run-time errors**.

Detecting Run-Time Errors

We can use various **analysis** techniques to detect run-time errors:



An **analyzer** is a program that takes a program as an input and determines whether the program has a certain property. In this case, the property is **run-time errors**.

We can categorize them into two groups:

- **Dynamic Analysis**: analyze programs by **executing** them
- **Static Analysis**: analyze programs **without executing** them

Dynamic Analysis

Dynamic analysis is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
}                        /* L6 */
```

Dynamic Analysis

Dynamic analysis is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```scala
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
}                        /* L6 */
```

L1	-5
L2	-5
L3	5
L4	
L5	
L6	5

Dynamic Analysis

Dynamic analysis is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```scala
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
}                        /* L6 */
```

L1	-5	42
L2	-5	
L3	5	
L4		42
L5		42
L6	5	42

Dynamic Analysis

Dynamic analysis is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```scala
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
}                        /* L6 */
```

L1	-5	42	-7	99	0	...
L2	-5		-7			...
L3	5		7			...
L4		42		99	0	...
L5		42		99	0	...
L6	5	42	7	99	0	...

Dynamic Analysis

Dynamic analysis is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
}                        /* L6 */
```

L1	-5	42	-7	99	0	...
L2	-5		-7			...
L3	5		7			...
L4		42		99	0	...
L5		42		99	0	...
L6	5	42	7	99	0	...

We can easily get the **behavior** of the program for each **single input**.

Dynamic Analysis

Dynamic analysis is a program analysis technique by **executing** them.

Let's perform **dynamic analysis** for the following Scala program:

```scala
def abs(x: Int): Int = { /* L1 */
  if (x < 0)            /* L2 */
    -x                  /* L3 */
  else                  /* L4 */
    x                   /* L5 */
}                       /* L6 */
```

L1	-5	42	-7	99	0	...
L2	-5		-7			...
L3	5		7			...
L4		42		99	0	...
L5		42		99	0	...
L6	5	42	7	99	0	...

We can easily get the **behavior** of the program for each **single input**.

However, it is **difficult** to get all the **possible behaviors** of the program for **all the inputs**.

Static Analysis

Static analysis is a program analysis technique **without executing** them.

Let's perform **static analysis** for the following Scala program:

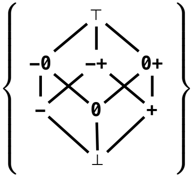
```scala
def abs(x: Int): Int = { /* L1 */
  if (x < 0)            /* L2 */
    -x                  /* L3 */
  else                  /* L4 */
    x                   /* L5 */
}                       /* L6 */
```

Static Analysis

Static analysis is a program analysis technique **without executing** them.

Let's perform **static analysis** for the following Scala program:

```scala
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
}                        /* L6 */
```

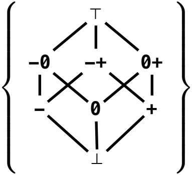
$\mathbb{D} =$ 

Let's define an **abstract domain** $\mathbb{D}$ for integers to analyze the program.

Static analysis is a program analysis technique **without executing** them.

Let's perform **static analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
}                        /* L6 */
```

$\mathbb{D} =$ 

Let's define an **abstract domain** $\mathbb{D}$ for integers to analyze the program.

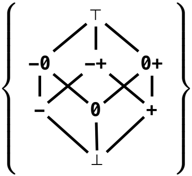
$$\begin{array}{lll} \bot = \varnothing & \top = \mathbb{Z} & \\ \texttt{0} = \{0\} & \texttt{-} = \{x \in \mathbb{Z} \mid x < 0\} & \texttt{+} = \{x \in \mathbb{Z} \mid x > 0\} \\ \texttt{-0} = \texttt{-} \cup \texttt{0} & \texttt{-+} = \texttt{-} \cup \texttt{+} & \texttt{0+} = \texttt{0} \cup \texttt{+} \end{array}$$

Static Analysis

Static analysis is a program analysis technique **without executing** them.

Let's perform **static analysis** for the following Scala program:

```scala
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
}                        /* L6 */
```

$\mathbb{D} =$ 

L1	
L2	
L3	
L4	
L5	
L6	

Let's define an **abstract domain** $\mathbb{D}$ for integers to analyze the program:

$\bot = \varnothing$ $\qquad$ $\top = \mathbb{Z}$

$\texttt{0} = \{0\}$ $\qquad$ $\texttt{-} = \{x \in \mathbb{Z} \mid x < 0\}$ $\qquad$ $\texttt{+} = \{x \in \mathbb{Z} \mid x > 0\}$

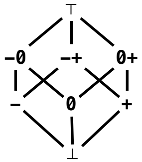
$\texttt{-0} = \texttt{-} \cup \texttt{0}$ $\qquad$ $\texttt{-+} = \texttt{-} \cup \texttt{+}$ $\qquad$ $\texttt{0+} = \texttt{0} \cup \texttt{+}$

Static Analysis

Static analysis is a program analysis technique **without executing** them.

Let's perform **static analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */
  if (x < 0)            /* L2 */
    -x                  /* L3 */
  else                  /* L4 */
    x                   /* L5 */
}                       /* L6 */
```

$\mathbb{D} = \{$  $\}$

L1	$\top$
L2	-
L3	+
L4	0+
L5	0+
L6	0+

Let's define an **abstract domain** $\mathbb{D}$ for integers to analyze the program:

$$\bot = \varnothing \qquad \top = \mathbb{Z}$$

$$0 = \{0\} \qquad \texttt{-} = \{x \in \mathbb{Z} \mid x < 0\} \qquad \texttt{+} = \{x \in \mathbb{Z} \mid x > 0\}$$

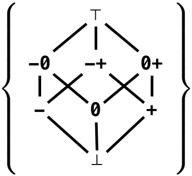
$$\texttt{-0} = \texttt{-} \cup 0 \qquad \texttt{-+} = \texttt{-} \cup \texttt{+} \qquad \texttt{0+} = 0 \cup \texttt{+}$$

Static Analysis

Static analysis is a program analysis technique **without executing** them.

Let's perform **static analysis** for the following Scala program:

```
def abs(x: Int): Int = { /* L1 */
  if (x < 0)             /* L2 */
    -x                   /* L3 */
  else                   /* L4 */
    x                    /* L5 */
}                        /* L6 */
```

$\mathbb{D} =$ 

L1	$\top$
L2	-
L3	+
L4	0+
L5	0+
L6	0+

Let's define an **abstract domain** $\mathbb{D}$ for integers to analyze the program:

$$
\begin{array}{lll}
\bot = \varnothing & \top = \mathbb{Z} & \\
0 = \{0\} & - = \{x \in \mathbb{Z} \mid x < 0\} & + = \{x \in \mathbb{Z} \mid x > 0\} \\
-0 = - \cup 0 & -+ = - \cup + & 0+ = 0 \cup +
\end{array}
$$

We can prove that `abs` always returns a **non-negative** integer (i.e., 0+).

Soundness vs Completeness

- $\vdash \psi$ denotes that a statement ψ is **provable**.
- $\models \psi$ denotes that a statement ψ is **true**.

In a **sound** proof system, all **provable** statements are **true**.

$$\vdash \psi \quad \Longrightarrow \quad \models \psi$$

In a **complete** proof system, all **true** statements are **provable**.

$$\models \psi \quad \Longrightarrow \quad \vdash \psi$$

Soundness vs Completeness

- $\vdash \psi$ denotes that a statement ψ is **provable**.
- $\vDash \psi$ denotes that a statement ψ is **true**.

In a **sound** proof system, all **provable** statements are **true**.

$$\vdash \psi \qquad \Longrightarrow \qquad \vDash \psi$$

In a **complete** proof system, all **true** statements are **provable**.

$$\vDash \psi \qquad \Longrightarrow \qquad \vdash \psi$$

Analysis techniques can be used to prove that a program is **error-free**.

- $\vdash P$ denotes that a program P is **analyzed** as error-free.
- $\vDash P$ denotes that a program P is truly **error-free**.

Then, is dynamic/static analysis **sound** or **complete**?

Soundness vs Completeness

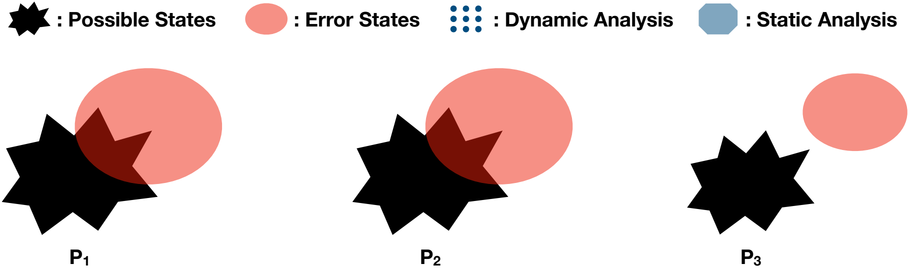
- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).

Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
  - **Not** all detected errors are **true alarms**.
  - It suffers from **false alarms** (**false positive (FP)**).
  - There are **no missing errors**. We can **prove** a program is error-free.

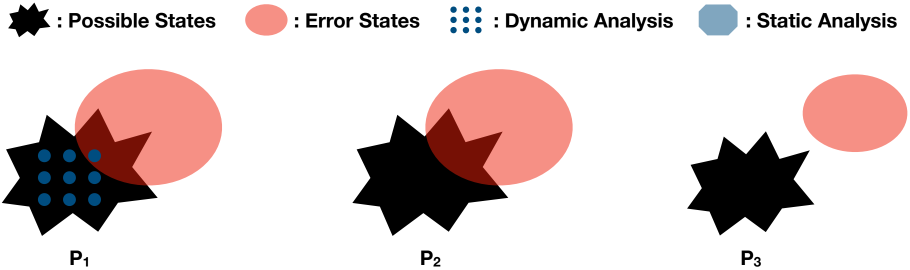
Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
  - **Not** all detected errors are **true alarms**.
  - It suffers from **false alarms** (**false positive (FP)**).
  - There are **no missing errors**. We can **prove** a program is error-free.



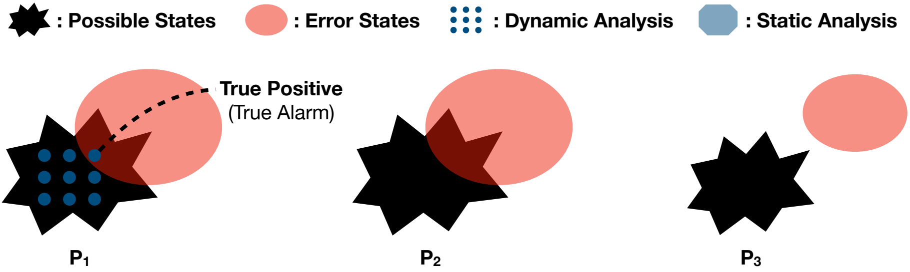
Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
  - **Not** all detected errors are **true alarms**.
  - It suffers from **false alarms** (**false positive (FP)**).
  - There are **no missing errors**. We can **prove** a program is error-free.



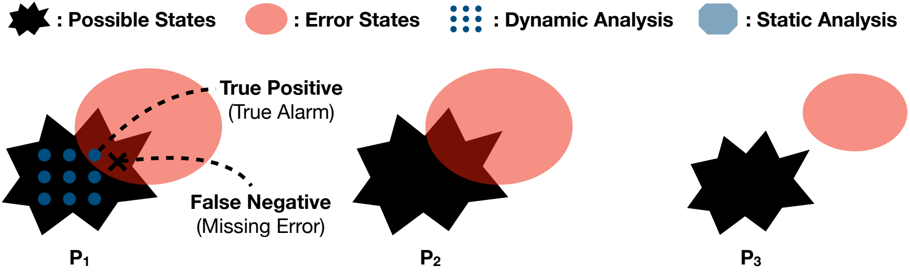
Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
  - **Not** all detected errors are **true alarms**.
  - It suffers from **false alarms** (**false positive (FP)**).
  - There are **no missing errors**. We can **prove** a program is error-free.



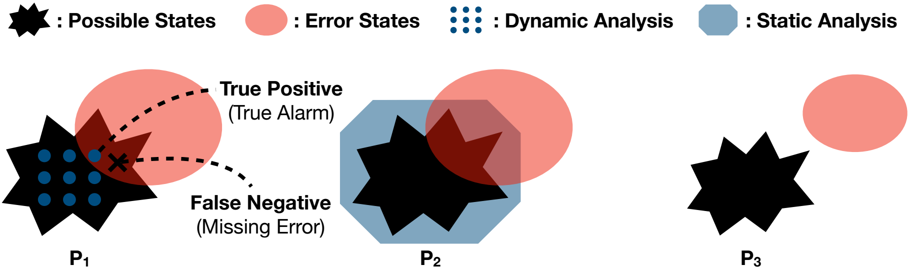
Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
  - **Not** all detected errors are **true alarms**.
  - It suffers from **false alarms** (**false positive (FP)**).
  - There are **no missing errors**. We can **prove** a program is error-free.



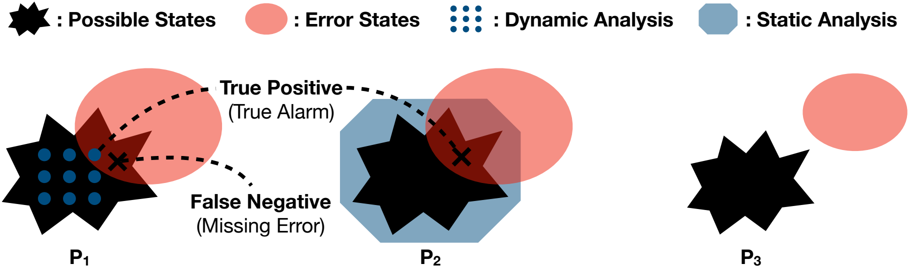
Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
  - **Not** all detected errors are **true alarms**.
  - It suffers from **false alarms** (**false positive (FP)**).
  - There are **no missing errors**. We can **prove** a program is error-free.



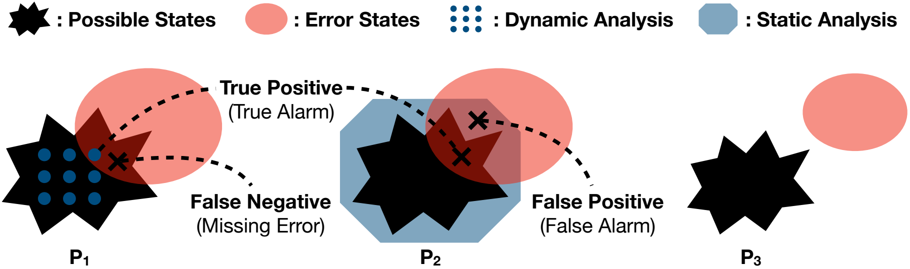
Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
  - **Not** all detected errors are **true alarms**.
  - It suffers from **false alarms** (**false positive (FP)**).
  - There are **no missing errors**. We can **prove** a program is error-free.



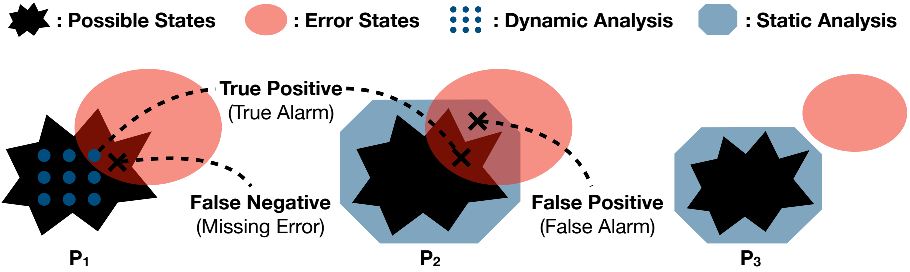
Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
  - **Not** all detected errors are **true alarms**.
  - It suffers from **false alarms** (**false positive (FP)**).
  - There are **no missing errors**. We can **prove** a program is error-free.



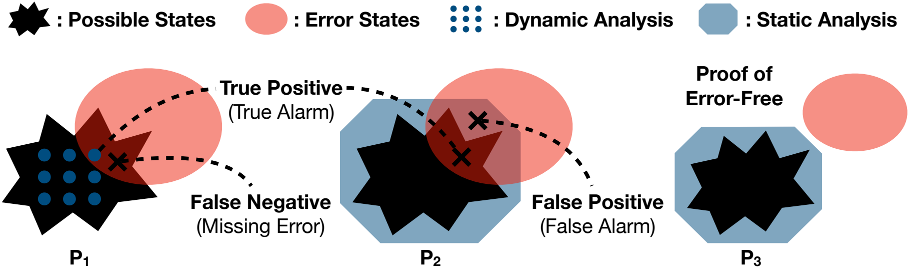
Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
  - **Not** all detected errors are **true alarms**.
  - It suffers from **false alarms** (**false positive (FP)**).
  - There are **no missing errors**. We can **prove** a program is error-free.



Soundness vs Completeness

- **Dynamic analysis** is **complete** but **unsound** in general.
  - All the detected errors are **true alarms** (**true positive (TP)**).
  - It will not detect any errors in error-free programs.
  - It suffers from **missing errors** (**false negative (FN)**).
- **Static analysis** is **sound** but **incomplete** in general.
  - **Not** all detected errors are **true alarms**.
  - It suffers from **false alarms** (**false positive (FP)**).
  - There are **no missing errors**. We can **prove** a program is error-free.



Contents

PLRG

Definition (Types)

A **type** is a set of values.

For example, the `Int`, `Boolean`, and `Int => Int` types are defined as the following sets of values in Scala.

$$
\begin{array}{lcl}
\texttt{Int} & = & \{n \in \mathbb{Z} \mid -2^{31} \leq n < 2^{31}\} \\
\texttt{Boolean} & = & \{\texttt{true}, \texttt{false}\} \\
\texttt{Int => Int} & = & \{f \mid f \text{ is a function from } \texttt{Int} \text{ to } \texttt{Int}\}
\end{array}
$$

```
val n: Int = 42              // 42   : Int
n + 1                        // 43   : Int
val b: Boolean = n > 10      // true : Boolean
def f(x: Int): Int = x + 1   // f    : Int => Int
f(42)                        // 43   : Int
```

Type Errors

PLRG

Definition (Type Errors)

A **type error** occurs when a program tries to use a value having a type that is **incompatible** with the expected type.

For example, the following Scala program has type errors:

```
42 + true               // `Int` expected for `+`, but `Boolean` found
if (1) 2 else 3         // `Boolean` expected for `if`, but `Int` found
def f(x: Int): Int = x + 1
f(false)                // `Int` expected for `f`, but `Boolean` found
```

However, not all **run-time errors** are **type errors**:

```
42 / 0                  // `ArithmeticException` at run-time
case class A(k: Int)
val x: A = null
x.k                     // `NullPointerException` at run-time
```

Type Checking

If the following conditions hold, we say **"the expression e has type τ"**:

- e does not cause any type error, and
- e evaluates to a value of type τ or does not terminate.

Type Checking

If the following conditions hold, we say **"the expression e has type τ"**:

- e does not cause any type error, and
- e evaluates to a value of type τ or does not terminate.

If so, we use the following notation and say that e is **well-typed**:

$$\boxed{\vdash e : \tau}$$

Type Checking

If the following conditions hold, we say **"the expression e has type τ"**:

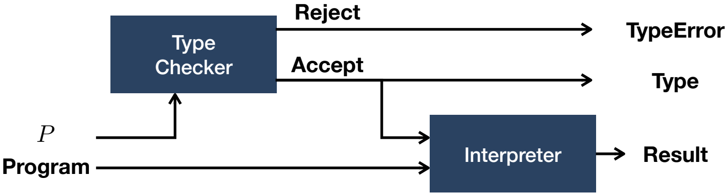
- e does not cause any type error, and
- e evaluates to a value of type τ or does not terminate.

If so, we use the following notation and say that e is **well-typed**:

$$\boxed{\vdash e : \tau}$$

Definition (Type Checking)

Type checking is a kind of static analysis checking whether a given expression e is **well-typed**. A **type checker** returns the **type** of e if it is well-typed, or rejects it and reports the detected **type error** otherwise.



Type Soundness

Definition (Type Soundness)

A **type system** is **sound** if it guarantees that a **well-typed** program will **never** cause a **type error** at run-time.

Type Soundness

Definition (Type Soundness)

A **type system** is **sound** if it guarantees that a **well-typed** program will **never** cause a **type error** at run-time.

There are two categories of languages in the context of type system:

- **Statically-typed languages** (or simply typed languages) only allow **well-typed** programs to be executed.
  (e.g. Java, Scala, Haskell, OCaml, Rust, etc.)

Type Soundness

Definition (Type Soundness)

A **type system** is **sound** if it guarantees that a **well-typed** program will **never** cause a **type error** at run-time.

There are two categories of languages in the context of type system:

- **Statically-typed languages** (or simply typed languages) only allow **well-typed** programs to be executed.
  (e.g. Java, Scala, Haskell, OCaml, Rust, etc.)
- **Dynamically-typed languages** (or simply untyped languages) allow any program to be executed, and types exist only at run-time.
  (e.g. Python, Ruby, JavaScript, etc.)

Type Soundness

Definition (Type Soundness)

A **type system** is **sound** if it guarantees that a **well-typed** program will **never** cause a **type error** at run-time.

There are two categories of languages in the context of type system:

- **Statically-typed languages** (or simply typed languages) only allow **well-typed** programs to be executed.
  (e.g. Java, Scala, Haskell, OCaml, Rust, etc.)
- **Dynamically-typed languages** (or simply untyped languages) allow any program to be executed, and types exist only at run-time.
  (e.g. Python, Ruby, JavaScript, etc.)

Type systems in most statically-typed languages are designed to be **sound**.

Summary

1. Motivation: Safe Language Systems
   - Detecting Run-Time Errors
   - Dynamic vs Static Analysis
   - Soundness vs Completeness

2. Type Systems
   - Types
   - Type Errors
   - Type Checking
   - Type Soundness

Next Lecture

- Typed Languages

Jihyeok Park
jihyeok_park@korea.ac.kr
https://plrg.korea.ac.kr