

Lecture 5 – Identifiers (2)

COSE212: Programming Languages

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- **Identifiers**
 - Bound identifiers
 - Free identifiers
 - Shadowing
- VAE – AE **with variables**
 - Concrete syntax
 - Abstract syntax

- **Identifiers**
 - Bound identifiers
 - Free identifiers
 - Shadowing
- VAE – AE **with variables**
 - Concrete syntax
 - Abstract syntax
- In this lecture, we will
 - implement the **interpreter** for VAE
 - define the **natural semantics** for VAE

1. Evaluation with Environments

2. Interpreter and Natural Semantics for VAE

- Numbers

- Addition and Multiplication

- Variable Definition

- Variable Lookup

3. Examples

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3. Examples

Let's evaluate the following VAE expressions:

```
/* VAE */  
val x = 1; {           // [ x -> 1 ]  
  val y = 2; {         // [ x -> 1, y -> 2 ]  
    x + y              // x + y = 1 + 2 = 3  
  }  
}
```

Let's evaluate the following VAE expressions:

```
/* VAE */  
val x = 1; {           // [ x -> 1 ]  
  val y = 2; {         // [ x -> 1, y -> 2 ]  
    x + y              // x + y = 1 + 2 = 3  
  }  
}
```

How to evaluate the expression $x + y$ into the value 3?

$$\vdash x + y \Rightarrow 3$$

Let's evaluate the following VAE expressions:

```
/* VAE */  
val x = 1; {           // [ x -> 1 ]  
  val y = 2; {         // [ x -> 1, y -> 2 ]  
    x + y              // x + y = 1 + 2 = 3  
  }  
}
```

How to evaluate the expression $x + y$ into the value 3?

$$\vdash x + y \Rightarrow 3$$

We need to keep track of the **environment** that maps identifiers to values:

$$[x \mapsto 1, y \mapsto 2] \vdash x + y \Rightarrow 3$$


```
type Value = BigInt           // values
def interp(expr: Expr): Value = ... // interpreter
```

For AE, the interpreter takes an expression and returns a number.

$$\vdash e \Rightarrow n$$

```
type Value = BigInt                                // values
type Env = Map[String, Value]                     // environments
def interp(expr: Expr, env: Env): Value = ... // interpreter
```

For VAE, we extend the interpreter to take an **environment** as well.

$$\sigma \vdash e \Rightarrow n$$

We read it as “*with the **environment** σ , the **expression** e evaluates to the **number** n* ”

```
type Value = BigInt                                // values
type Env = Map[String, Value]                     // environments
def interp(expr: Expr, env: Env): Value = ...    // interpreter
```

For VAE, we extend the interpreter to take an **environment** as well.

$$\sigma \vdash e \Rightarrow n$$

We read it as “*with the **environment** σ , the **expression** e evaluates to the **number** n* ”

For example, the interpreter should be able to evaluate like this:

```
val env : Env    = Map("x" -> 1, "y" -> 2)    // [ x -> 1, y -> 2 ]
val expr: Expr   = Expr("x + y")              // Add(Id("x"), Id("y"))
val v  : Value   = interp(expr, env)           // 3
```

$$[x \mapsto 1, y \mapsto 2] \vdash x + y \Rightarrow 3$$

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For VAE, we need to 1) implement the **interpreter** with **environments**

```
def interp(expr: Expr, env: Env): Value = ???
```

For VAE, we need to 1) implement the **interpreter** with **environments**

```
def interp(expr: Expr, env: Env): Value = ???
```

and 2) define the **natural semantics** with **environments**.

$$\sigma \vdash e \Rightarrow n$$

Expressions	$e ::= n$	(Num)
	$e + e$	(Add)
	$e * e$	(Mul)
	$\text{val } x = e; e$	(Val)
	x	(Id)

where

Environments	$\sigma \in \mathbb{X} \xrightarrow{\text{fin}} \mathbb{Z}$	(Env)
Numbers	$n \in \mathbb{Z}$	(BigInt)
Identifiers	$x \in \mathbb{X}$	(String)

```
def interp(expr: Expr, env: Env): Value = expr match
  case Num(n)      => ???
  case Add(l, r)    => ???
  case Mul(l, r)    => ???
  case Val(x, e, b) => ???
  case Id(x)        => ???
```

$$\boxed{\sigma \vdash e \Rightarrow n}$$

$$\text{NUM} \frac{\text{???}}{\sigma \vdash n \Rightarrow \text{???}}$$

$$\text{ADD} \frac{\text{???}}{\sigma \vdash e_1 + e_2 \Rightarrow \text{???}}$$

$$\text{MUL} \frac{\text{???}}{\sigma \vdash e_1 * e_2 \Rightarrow \text{???}}$$

$$\text{VAL} \frac{\text{???}}{\sigma \vdash \text{val } x = e_1; e_2 \Rightarrow \text{???}}$$

$$\text{ID} \frac{\text{???}}{\sigma \vdash x \Rightarrow \text{???}}$$

```
def interp(expr: Expr, env: Env): Value = expr match
  case Num(n)      => ???
  ...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{NUM} \frac{\text{???}}{\sigma \vdash n \Rightarrow \text{???}}$$


```
def interp(expr: Expr, env: Env): Value = expr match
  case Num(n)      => n
  ...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{NUM} \frac{}{\sigma \vdash n \Rightarrow n}$$

With the **environment** σ , the **expression** n evaluates to the **number** n .

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Add(l, r)    => ???
  ...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{ADD} \frac{\text{???}}{\sigma \vdash e_1 + e_2 \Rightarrow \text{???}}$$

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Add(l, r)    => interp(l, env) + interp(r, env)
...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{ADD} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 + e_2 \Rightarrow n_1 + n_2}$$

With the **environment** σ , the **expression** $e_1 + e_2$ evaluates to the **number** $n_1 + n_2$ when

- With the **environment** σ , the **expression** e_1 evaluates to the **number** n_1 .
- With the **environment** σ , the **expression** e_2 evaluates to the **number** n_2 .

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Mul(l, r)    => interp(l, env) * interp(r, env)
...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{MUL} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 * e_2 \Rightarrow n_1 \times n_2}$$

With the **environment** σ , the **expression** $e_1 * e_2$ evaluates to the **number** $n_1 \times n_2$ when

- With the **environment** σ , the **expression** e_1 evaluates to the **number** n_1 .
- With the **environment** σ , the **expression** e_2 evaluates to the **number** n_2 .

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Val(x, e, b) => ???
  ...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{VAL} \frac{\text{???}}{\sigma \vdash \text{val } x = e_1; e_2 \Rightarrow \text{???}}$$

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Val(x, e, b) => ... interp(e, env) ...
  ...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{VAL} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \dots}{\sigma \vdash \text{val } x = e_1; e_2 \Rightarrow ???}$$

With the **environment** σ , the **expression** $\text{val } x = e_1; e_2$ evaluates to the **number** $???$ when

- ① With the **environment** σ , the **expression** e_1 evaluates to the **number** n_1 .
- ② ...

```
def interp(expr: Expr, env: Env): Value = expr match
...
case Val(x, e, b) => ... env + (x -> interp(e, env)) ...
...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{VAL} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma[x \mapsto n_1] \quad \dots}{\sigma \vdash \text{val } x = e_1; e_2 \Rightarrow ???}$$

With the **environment** σ , the **expression** $\text{val } x = e_1; e_2$ evaluates to the **number** $???$ when

- ① With the **environment** σ , the **expression** e_1 evaluates to the **number** n_1 .
- ② With the **environment** $\sigma[x \mapsto n_1], \dots$

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Val(x, e, b) => interp(b, env + (x -> interp(e, env)))
  ...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{VAL} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma[x \mapsto n_1] \vdash e_2 \Rightarrow n_2}{\sigma \vdash \text{val } x = e_1; e_2 \Rightarrow n_2}$$

With the **environment** σ , the **expression** $\text{val } x = e_1; e_2$ evaluates to the **number** n_2 when

- ① With the **environment** σ , the **expression** e_1 evaluates to the **number** n_1 .
- ② With the **environment** $\sigma[x \mapsto n_1]$, the **expression** e_2 evaluates to the **number** n_2 .


```
def interp(expr: Expr, env: Env): Value = expr match
...
case Id(x)          => ???
...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{ID} \frac{\text{???}}{\sigma \vdash x \Rightarrow \text{???}}$$

```
def interp(expr: Expr, env: Env): Value = expr match
  ...
  case Id(x)          => env.getOrElse(x, error(s"free identifier: $x"))
  ...
```

$$\sigma \vdash e \Rightarrow n$$

$$\text{ID} \frac{x \in \text{dom}(\sigma)}{\sigma \vdash x \Rightarrow \sigma(x)}$$

With the **environment** σ , the **expression** x evaluates to the **number** $\sigma(x)$ when

- 1 The **variable** x is in the domain of the **environment** σ .

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Example 1

$$\frac{\text{NUM} \frac{}{\emptyset \vdash 1 \Rightarrow 1} \quad \text{ADD} \frac{\text{ID} \frac{x \in \text{dom}([x \mapsto 1])}{[x \mapsto 1] \vdash x \Rightarrow 1} \quad \text{NUM} \frac{}{[x \mapsto 1] \vdash 2 \Rightarrow 2}}{[x \mapsto 1] \vdash x + 2 \Rightarrow 3}}{\text{VAL} \frac{}{\emptyset \vdash \text{val } x = 1; x + 2 \Rightarrow 3}}$$

Example 1

$$\begin{array}{c}
 \text{ID} \frac{x \in \text{dom}([x \mapsto 1])}{[x \mapsto 1] \vdash x \Rightarrow 1} \quad \text{NUM} \frac{}{[x \mapsto 1] \vdash 2 \Rightarrow 2} \\
 \text{ADD} \frac{}{[x \mapsto 1] \vdash x + 2 \Rightarrow 3} \\
 \text{NUM} \frac{}{\emptyset \vdash 1 \Rightarrow 1} \\
 \text{VAL} \frac{}{\emptyset \vdash \text{val } x = 1; x + 2 \Rightarrow 3}
 \end{array}$$

We can name environments σ_i to make the derivation tree concise.

$$\begin{array}{c}
 \text{ID} \frac{x \in \text{dom}(\sigma_0)}{\sigma_0 \vdash x \Rightarrow 1} \quad \text{NUM} \frac{}{\sigma_0 \vdash 2 \Rightarrow 2} \\
 \text{ADD} \frac{}{\sigma_0 \vdash x + 2 \Rightarrow 3} \\
 \text{NUM} \frac{}{\emptyset \vdash 1 \Rightarrow 1} \\
 \text{VAL} \frac{}{\emptyset \vdash \text{val } x = 1; x + 2 \Rightarrow 3}
 \end{array}$$

where

$$\sigma_0 = [x \mapsto 1]$$

Example 2

VAL $\frac{}{\emptyset \vdash \text{val } x = 1; \{\text{val } y = 2; x + y\} \Rightarrow}$

where

$$\begin{array}{c}
 \text{NUM} \frac{}{\emptyset \vdash 1 \Rightarrow 1} \quad \text{VAL} \frac{\text{NUM} \frac{}{\sigma_0 \vdash 2 \Rightarrow 2} \quad \text{ADD} \frac{\text{ID} \frac{x \in \text{dom}(\sigma_1)}{\sigma_1 \vdash x \Rightarrow 1} \quad \text{ID} \frac{y \in \text{dom}(\sigma_1)}{\sigma_1 \vdash y \Rightarrow 2}}{\sigma_1 \vdash x + y \Rightarrow 3}}{\sigma_0 \vdash \text{val } y = 2; x + y \Rightarrow 3}} \\
 \text{VAL} \frac{}{\emptyset \vdash \text{val } x = 1; \{\text{val } y = 2; x + y\} \Rightarrow 3}
 \end{array}$$

where

$$\begin{aligned}
 \sigma_0 &= [x \mapsto 1] \\
 \sigma_1 &= [x \mapsto 1, y \mapsto 2]
 \end{aligned}$$

Example 3

VAL $\frac{}{\emptyset \vdash \text{val } x = 1; \{ \text{val } x = 2; x \} + x \Rightarrow}$

where

$$\begin{array}{c}
 \text{NUM} \frac{}{\emptyset \vdash 1 \Rightarrow 1} \quad \text{VAL} \frac{\text{NUM} \frac{}{\sigma_0 \vdash 2 \Rightarrow 2} \quad \text{ID} \frac{x \in \text{dom}(\sigma_1)}{\sigma_1 \vdash x \Rightarrow 2}}{\sigma_0 \vdash \text{val } x = 2; x \Rightarrow 2} \quad \text{ID} \frac{x \in \text{dom}(\sigma_0)}{\sigma_0 \vdash x \Rightarrow 1} \\
 \text{ADD} \frac{}{\sigma_0 \vdash \{\text{val } x = 2; x\} + x \Rightarrow 3} \\
 \text{VAL} \frac{}{\emptyset \vdash \text{val } x = 1; \{\text{val } x = 2; x\} + x \Rightarrow 3}
 \end{array}$$

where

$$\begin{aligned}
 \sigma_0 &= [x \mapsto 1] \\
 \sigma_1 &= [x \mapsto 2]
 \end{aligned}$$

$$\begin{array}{c}
 \text{NUM} \frac{}{\emptyset \vdash 1 \Rightarrow 1} \quad \text{ID} \frac{x \in \text{dom}(\sigma_0)}{\sigma_0 \vdash x \Rightarrow 1} \\
 \text{VAL} \frac{}{\emptyset \vdash \text{val } x = 1; x \Rightarrow 1} \quad \text{ID} \frac{x \notin \text{dom}(\emptyset)}{\emptyset \vdash x \Rightarrow \text{FAIL}} \\
 \text{ADD} \frac{}{\emptyset \vdash \{\text{val } x = 1; x\} + x \Rightarrow \text{FAIL}}
 \end{array}$$

where

$$\sigma_0 = [x \mapsto 1]$$

We cannot draw the derivation tree for this example because of the **free variable** x in the right-hand side of the addition. The **red** part above is not a real derivation: it marks where **no rule applies**, since the Id rule requires $x \in \text{dom}(\emptyset)$, and **FAIL** merely denotes that the evaluation gets **stuck** there.

```
def interp(expr: Expr, env: Env): Value = expr match
  case Num(n)          => n
  case Add(l, r)        => interp(l, env) + interp(r, env)
  case Mul(l, r)        => interp(l, env) * interp(r, env)
  case Val(x, e, b)     => interp(b, env + (x -> interp(e, env)))
  case Id(x)            => env.getOrElse(x, error(s"free identifier: $x"))
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$$\boxed{\sigma \vdash e \Rightarrow n}$$

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$$\text{MUL} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 * e_2 \Rightarrow n_1 \times n_2}$$

$$\text{VAL} \frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma[x \mapsto n_1] \vdash e_2 \Rightarrow n_2}{\sigma \vdash \text{val } x = e_1; e_2 \Rightarrow n_2}$$

$$\text{ID} \frac{x \in \text{dom}(\sigma)}{\sigma \vdash x \Rightarrow \sigma(x)}$$

<https://github.com/ku-plrg-classroom/docs/tree/main/cose212/vae>

- Please see above document on GitHub:
 - Implement `interp` function.
 - Implement `freeIds` function.
 - Implement `bindingIds` function.
 - Implement `boundIds` function.
 - Implement `shadowedIds` function.
- It is just an exercise, and you **don't need to submit** anything.
- However, some exam questions might be related to this exercise, so we **strongly recommend** solving it by yourself.

- First-Order Functions

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