Lecture 16 – Equivalence of Pushdown Automata and Context-Free Grammars

COSE215: Theory of Computation

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PLRG

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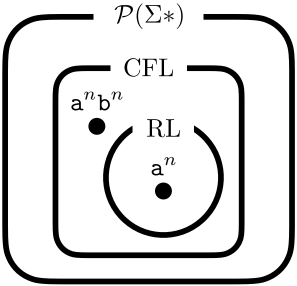
Recall

A **context-free grammar** is a 4-tuple:

$$G = (V, \Sigma, S, R)$$

A **pushdown automaton (PDA)** is a finite automaton with a **stack**.

- Acceptance by **final states**
- Acceptance by **empty stacks**



PDA_{FS}
(by final states)

$=$ ----- ?

PDA_{ES}
(by empty stacks)

$=$ ----- ?

CFG

Contents

1. Equivalence of PDA by Final States and Empty Stacks
   - $\mathrm{PDA}_{\mathrm{FS}}$ to $\mathrm{PDA}_{\mathrm{ES}}$
   - $\mathrm{PDA}_{\mathrm{ES}}$ to $\mathrm{PDA}_{\mathrm{FS}}$

2. Equivalence of PDA and CFGs
   - CFGs to $\mathrm{PDA}_{\mathrm{ES}}$
   - $\mathrm{PDA}_{\mathrm{ES}}$ to CFGs

$$\underset{\text{(by final states)}}{\mathrm{PDA}_{\mathrm{FS}}} \rightleftarrows \underset{\text{(by empty stacks)}}{\mathrm{PDA}_{\mathrm{ES}}} \rightleftarrows \mathrm{CFG}$$

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$$\underset{\text{(by final states)}}{\text{PDA}_{\text{FS}}} \rightleftarrows \underset{\text{(by empty stacks)}}{\text{PDA}_{\text{ES}}} \rightleftarrows \text{CFG}$$

PDA_{FS} to PDA_{ES}

Theorem (PDA_{FS} to PDA_{ES})

For a given PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, $\exists$ *PDA* P'. $L_F(P) = L_E(P')$.

PDA_{FS} to PDA_{ES}

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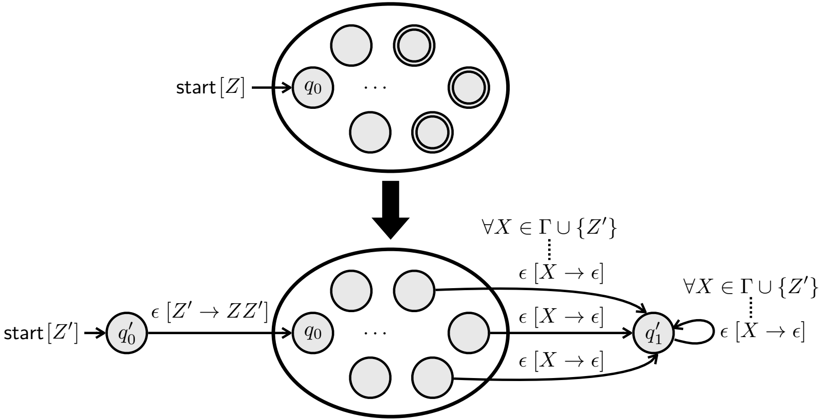


PDA_{FS} to PDA_{ES}

Theorem (PDA_{FS} to PDA_{ES})

For a given PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, $\exists$ PDA P'. $L_F(P) = L_E(P')$.

Define a PDA

$$P' = (Q \cup \{q_0', q_1'\}, \Sigma, \Gamma \cup \{Z'\}, \delta', q_0', Z', \varnothing)$$

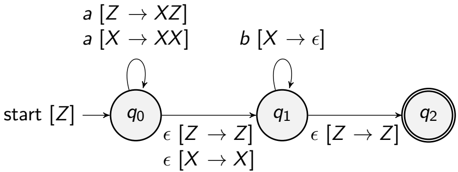
where

$$\begin{array}{lcl}
\delta'(q_0', \epsilon, Z') & = & \{(q_0, ZZ')\} \\
\delta'(q \in Q, a \in \Sigma, X \in \Gamma) & = & \delta(q, a, X) \\
\delta'(q \in Q, \epsilon, X \in \Gamma \cup \{Z'\}) & = & \left\{ \begin{array}{ll} \delta(q, \epsilon, X) \cup \{(q_1', \epsilon)\} & \text{if } q \in F \\ \delta(q, \epsilon, X) & \text{otherwise} \end{array} \right. \\
\delta'(q_1', \epsilon, X \in \Gamma \cup \{Z'\}) & = & \{(q_1', \epsilon)\}
\end{array}$$

PDA_{FS} to PDA_{ES} – Example

$$L_F(P) = L_E(P') = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$$

$P =$



PDA$_{FS}$ to PDA$_{ES}$ – Example

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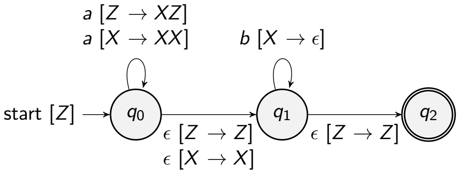
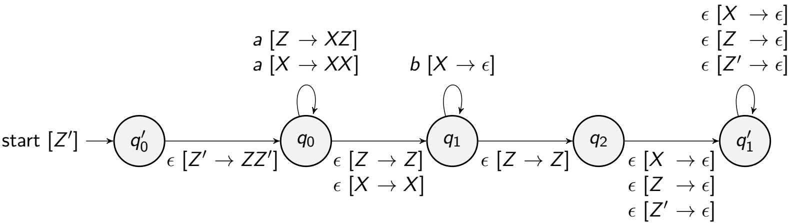


$P' =$



PDA_{ES} to PDA_{FS}

Theorem (PDA_{ES} to PDA_{FS})

For a given PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, $\exists$ *PDA* P'. $L_E(P) = L_F(P')$.

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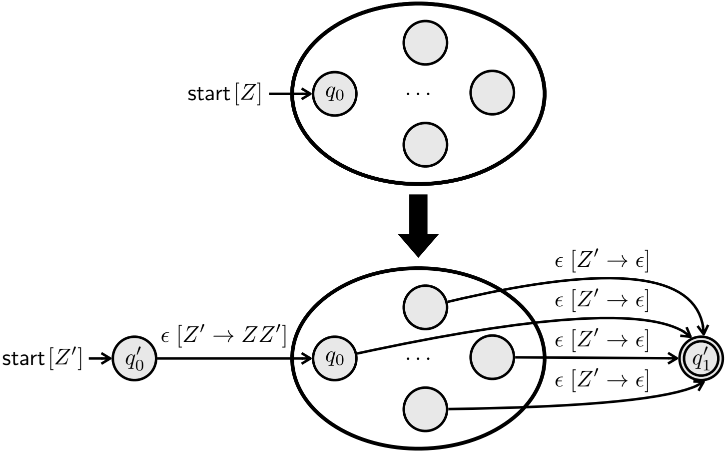

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Theorem (PDA_{ES} to PDA_{FS})

For a given PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, $\exists$ *PDA* P'. $L_E(P) = L_F(P')$.

Define a PDA

$$P' = (Q \cup \{q'_0, q'_1\}, \Sigma, \Gamma \cup \{Z'\}, \delta', q'_0, Z', \{q'_1\})$$

where

$$\begin{array}{lcl}
\delta'(q'_0, \epsilon, Z') & = & \{(q_0, ZZ')\} \\
\delta'(q \in Q, a \in \Sigma, X \in \Gamma) & = & \delta(q, a, X) \\
\delta'(q \in Q, \epsilon, X \in \Gamma) & = & \delta(q, \epsilon, X) \\
\delta'(q \in Q, \epsilon, Z') & = & \{(q'_1, \epsilon)\}
\end{array}$$

PDA_{ES} to PDA_{FS} – Example

$$L_E(P) = L_F(P') = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$$

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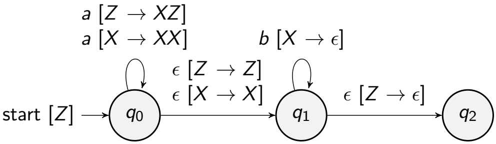

PDA$_{\text{ES}}$ to PDA$_{\text{FS}}$ – Example

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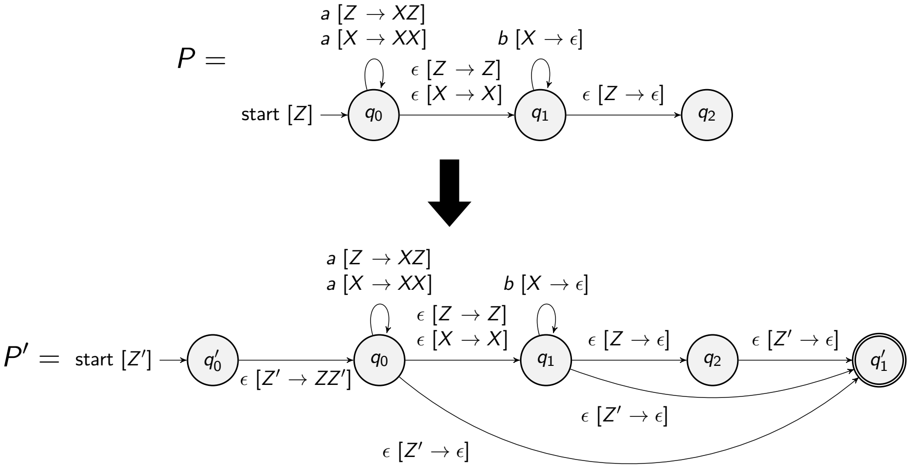

Contents

$$\underset{\text{(by final states)}}{\text{PDA}_{\text{FS}}} \rightleftarrows \underset{\text{(by empty stacks)}}{\text{PDA}_{\text{ES}}} \rightleftarrows \text{CFG}$$

CFGs to PDA_{ES}

Theorem (CFGs to PDA_{ES})

For a given CFG $G = (V, \Sigma, S, R)$, $\exists$ PDA P. $L(G) = L_E(P)$.

Define a PDA

$$P = (\{q\}, \Sigma, V \cup \Sigma, \delta, q, S, \varnothing)$$

where

$$\begin{array}{rcl} \delta(q, \epsilon, A \in V) & = & \{(q, \alpha) \mid A \to \alpha \in R\} \\ \delta(q, a \in \Sigma, a \in \Sigma) & = & \{(q, \epsilon)\} \end{array}$$

CFGs to PDA$_{\text{ES}}$ – Example

$$
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\delta(q, \epsilon, A \in V) & = & \{(q, \alpha) \mid A \to \alpha \in R\} \\
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\end{array}
$$

Consider the following CFG:

$$S \to \epsilon \mid \mathtt{a}S\mathtt{b} \mid \mathtt{b}S\mathtt{a} \mid SS$$

CFGs to PDA$_{\text{ES}}$ – Example

$$
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\delta(q, \epsilon, A \in V) &= \{(q, \alpha) \mid A \to \alpha \in R\} \\
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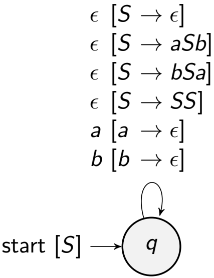
Then, the equivalent PDA (by empty stacks) is:



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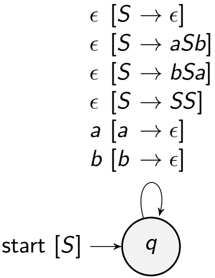

$$
\begin{aligned}
(q, \texttt{abab}, S) &\vdash (q, \texttt{abab}, \texttt{a}S\texttt{b}) \\
&\vdash (q, \texttt{bab}, S\texttt{b}) \\
&\vdash (q, \texttt{bab}, \texttt{b}S\texttt{ab}) \\
&\vdash (q, \texttt{ab}, S\texttt{ab}) \\
&\vdash (q, \texttt{ab}, \texttt{ab}) \\
&\vdash (q, \texttt{b}, \texttt{b}) \\
&\vdash (q, \epsilon, \epsilon)
\end{aligned}
$$

PDA_{ES} to CFGs

Theorem (PDA_{ES} to CFGs)

For a given PDA $P = (Q = \{q_0, \cdots, q_{n-1}\}, \Sigma, \Gamma, \delta, q_0, Z, F)$,
$\exists$ *CFG* G. $L_E(P) = L(G)$.

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Consider the set of variables $V = \{S\} \cup \{\ A^X_{i,j} \mid 0 \leq i, j < n \wedge X \in \Gamma\}$.

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Consider the set of variables $V = \{S\} \cup \{\ A^X_{i,j} \mid 0 \leq i, j < n \wedge X \in \Gamma\}$.
Then, define a CFG:

- For all $0 \leq j < n$,

$$S \to A^Z_{0,j}$$

- For all $q_i \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$, consider any $(q_j, X_1 \cdots X_m) \in \delta(q_i, a, X)$ and $0 \leq k_1, \cdots, k_m < n$. Then,

$$A^X_{i,k_m} \to a\ A^{X_1}_{j,k_1}\ A^{X_2}_{k_1,k_2} \cdots A^{X_m}_{k_{m-1},k_m}$$

PDA_{ES} to CFGs

Theorem (PDA_{ES} to CFGs)

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- For all $0 \leq j < n$,
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$$A^X_{i,k_m} \to a\ A^{X_1}_{j,k_1}\ A^{X_2}_{k_1,k_2} \cdots A^{X_m}_{k_{m-1},k_m}$$

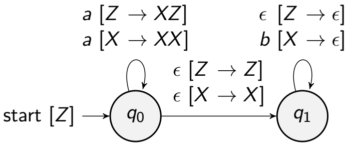
Note that each variable $A^X_{i,j}$ generates all words that cause the PDA to go from state q_i to state q_j by popping X:

$$A^X_{i,j} \Rightarrow^* w \qquad \text{if and only if} \qquad (q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$$

PDA_{ES} to CFGs – Example

$$S \to A^{Z}_{0,j} \qquad\qquad A^{X}_{i,k_m} \to a\ A^{X_1}_{j,k_1}\ A^{X_2}_{k_1,k_2} \cdots A^{X_m}_{k_{m-1},k_m}$$

Consider the following PDA (by empty stacks):



PDA_{ES} to CFGs – Example

$$S \rightarrow A^{Z}_{0,j} \qquad\qquad A^{X}_{i,k_m} \rightarrow a\ A^{X_1}_{j,k_1}\ A^{X_2}_{k_1,k_2} \cdots A^{X_m}_{k_{m-1},k_m}$$

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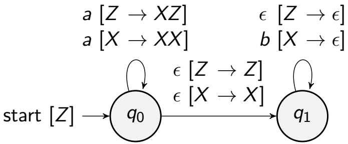

Then, the equivalent CFG is:

$$\begin{array}{lcl}
S & \rightarrow & A^{Z}_{0,0} \mid A^{Z}_{0,1} \\
A^{Z}_{0,0} & \rightarrow & \mathtt{a}\ A^{X}_{0,0}\ A^{Z}_{0,0} \mid \mathtt{a}\ A^{X}_{0,1}\ A^{Z}_{1,0} \mid A^{Z}_{1,0} \\
A^{Z}_{0,1} & \rightarrow & \mathtt{a}\ A^{X}_{0,0}\ A^{Z}_{0,1} \mid \mathtt{a}\ A^{X}_{0,1}\ A^{Z}_{1,1} \mid A^{Z}_{1,1} \\
A^{X}_{0,0} & \rightarrow & \mathtt{a}\ A^{X}_{0,0}\ A^{X}_{0,0} \mid \mathtt{a}\ A^{X}_{0,1}\ A^{X}_{1,0} \mid A^{X}_{1,0} \\
A^{X}_{0,1} & \rightarrow & \mathtt{a}\ A^{X}_{0,0}\ A^{X}_{0,1} \mid \mathtt{a}\ A^{X}_{0,1}\ A^{X}_{1,1} \mid A^{X}_{1,1} \\
A^{Z}_{1,1} & \rightarrow & \epsilon \\
A^{X}_{1,1} & \rightarrow & b
\end{array}$$

PDA_{ES} to CFGs – Example

$$S \to A^{Z}_{0,j} \qquad\qquad A^{X}_{i,k_m} \to a\ A^{X_1}_{j,k_1}\ A^{X_2}_{k_1,k_2} \cdots A^{X_m}_{k_{m-1},k_m}$$

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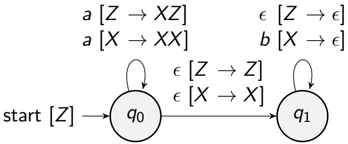

Then, the equivalent CFG is:

$$
\begin{array}{lcl}
S & \to & A^{Z}_{0,0} \mid A^{Z}_{0,1} \\
A^{Z}_{0,0} & \to & \text{a}\ A^{X}_{0,0}\ A^{Z}_{0,0} \mid \text{a}\ A^{X}_{0,1}\ A^{Z}_{1,0} \mid A^{Z}_{1,0} \\
A^{Z}_{0,1} & \to & \text{a}\ A^{X}_{0,0}\ A^{Z}_{0,1} \mid \text{a}\ A^{X}_{0,1}\ A^{Z}_{1,1} \mid A^{Z}_{1,1} \\
A^{X}_{0,0} & \to & \text{a}\ A^{X}_{0,0}\ A^{X}_{0,0} \mid \text{a}\ A^{X}_{0,1}\ A^{X}_{1,0} \mid A^{X}_{1,0} \\
A^{X}_{0,1} & \to & \text{a}\ A^{X}_{0,0}\ A^{X}_{0,1} \mid \text{a}\ A^{X}_{0,1}\ A^{X}_{1,1} \mid A^{X}_{1,1} \\
A^{Z}_{1,1} & \to & \epsilon \\
A^{X}_{1,1} & \to & b
\end{array}
$$

$$
\begin{array}{lcl}
S & \Rightarrow & A^{Z}_{0,1} \\
 & \Rightarrow & \text{a}\ A^{X}_{0,1}\ A^{Z}_{1,1} \\
 & \Rightarrow & \text{aa}\ A^{X}_{0,1}\ A^{X}_{1,1}\ A^{Z}_{1,1} \\
 & \Rightarrow & \text{aa}\ A^{X}_{1,1}\ A^{X}_{1,1}\ A^{Z}_{1,1} \\
 & \Rightarrow & \text{aab}\ A^{X}_{1,1}\ A^{Z}_{1,1} \\
 & \Rightarrow & \text{aabb}\ A^{Z}_{1,1} \\
 & \Rightarrow & \text{aabb}
\end{array}
$$

Summary

1. Equivalence of PDA by Final States and Empty Stacks
   - PDA_{FS} to PDA_{ES}
   - PDA_{ES} to PDA_{FS}

2. Equivalence of PDA and CFGs
   - CFGs to PDA_{ES}
   - PDA_{ES} to CFGs

$$\underset{\text{(by final states)}}{\mathrm{PDA}_{\mathrm{FS}}} \rightleftarrows \underset{\text{(by empty stacks)}}{\mathrm{PDA}_{\mathrm{ES}}} \rightleftarrows \mathrm{CFG}$$

Next Lecture

- Deterministic Pushdown Automata (DPDA)

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