Lecture 17 – Deterministic Pushdown Automata (DPDA)
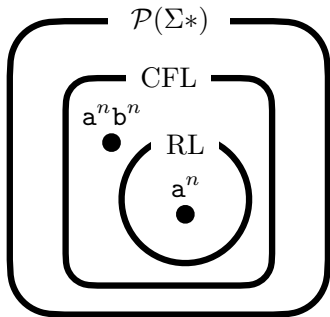## COSE215: Theory of Computation

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PLRG

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- A **pushdown automaton (PDA)** is an extension of ϵ-NFA with a **stack**. Thus, PDA is **non-deterministic**.
  - Acceptance by **final states**
  - Acceptance by **empty stacks**
- Then, how about **deterministic PDA (DPDA)**?
- What is the **language class** of DPDA? Still, CFL?

$\mathcal{P}(\Sigma*)$

CFL

$\mathtt{a}^n\mathtt{b}^n$ •

RL

$\mathtt{a}^n$ •

$\mathrm{PDA_{FS}}$
(by final states)

‖

$\mathrm{PDA_{ES}}$
(by empty stacks)

‖

CFG

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Definition (Deterministic Pushdown Automata)

A PDA is **deterministic** if there is at most one-step move ($\vdash$) from any configuration and we call it a **deterministic pushdown automaton (DPDA)**. In other words, a DPDA satisfies the following conditions:

- $|\delta(q, a, X)| \leq 1$ for all $q \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$.
- If $\delta(q, \epsilon, X) \neq \varnothing$, then $\delta(q, a, X) = \varnothing$ for all $a \in \Sigma$.

For example, the following PDA is **NOT** deterministic:
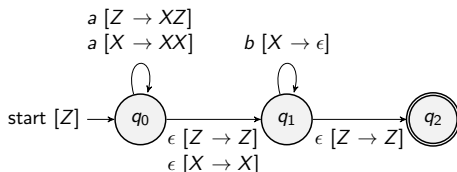
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For example, the following PDA is deterministic:
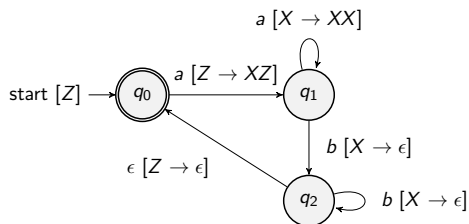
$$
\begin{aligned}
(q_0, \mathtt{aabb}, Z) &\vdash (q_1, \mathtt{abb}, XZ) \\
&\vdash (q_1, \mathtt{bb}, XXZ) \\
&\vdash (q_2, \mathtt{b}, XZ) \\
&\vdash (q_2, \epsilon, Z) \\
&\vdash (q_0, \epsilon, \epsilon)
\end{aligned}
$$

# Deterministic Context-Free Languages (DCFLs)

Definition (Deterministic Context-Free Languages (DCFLs))

A language L is a **deterministic context-free language (DCFL)** if and only if there exists a DPDA P such that $L = L_F(P)$ where $L_F(P)$ is the language accepted by **final states** of P.

For example, the following language is a DCFL:

$$L = \{a^n b^n \mid n \geq 0\}$$

because it is accepted by final states of the following DPDA:
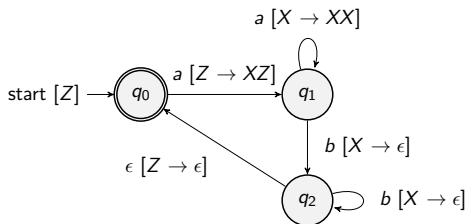
Fact 1: DCFL ⊂ CFL

Fact 1: DCFL ⊂ CFL

All DCFLs are CFLs **BUT** there exists a CFL that is not a DCFL.

For example, the following language is a CFL but not a DCFL:

$$L = \{ww^R \mid w \in \{a, b\}^*\}$$

Why?

Fact 2: RL ⊂ DCFL

All RLs are DCFLs **BUT** there exists a DCFL that is not an RL.

- $\boxed{\text{RL} \subseteq \text{DCFL}}$: For a given RL L and its corresponding DFA

$$D = (Q, \Sigma, \delta, q_0, F)$$

we can construct a DPDA P that accepts L as follows:

$$P = (Q, \Sigma, \{Z\}, \delta_P, q_0, Z, F)$$

where $\forall q \in Q. \ \forall a \in \Sigma. \ \delta_P(q, a, Z) = \{(\delta(q, a), Z)\}$. Then,

$$(q_0, w, Z) \vdash^* (q, \epsilon, Z) \quad \Longleftrightarrow \quad \delta^*(q_0, w) = q \qquad \square$$

- $\boxed{\text{DCFL} \backslash \text{RL} \neq \varnothing}$: We already know that the following language is a DCFL but not an RL:

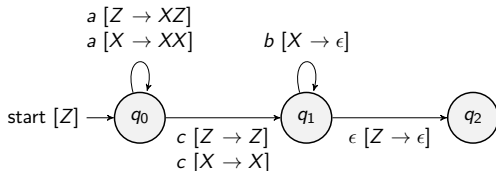$$L = \{a^n b^n \mid n \geq 0\} \in \text{DCFL} \setminus \text{RL}$$

Definition (DCFL$_{ES}$)

A language L is a **deterministic context-free language by empty stacks (DCFL$_{ES}$)** if and only if there exists a DPDA P such that $L = L_E(P)$ where $L_E(P)$ is the language accepted by **empty stacks** of P.

For example, the following language is a DCFL$_{ES}$:

$$L = \{a^n c b^n \mid n \geq 0\}$$
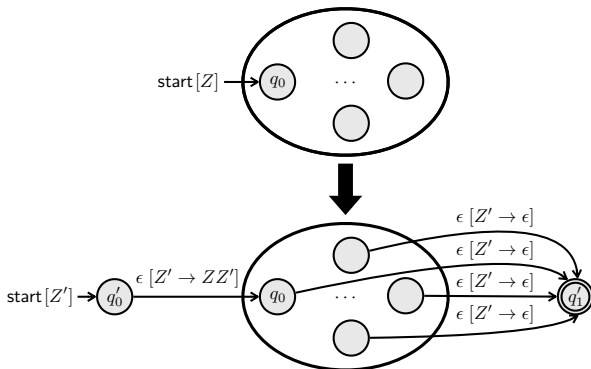
because it is accepted by empty stacks of the following DPDA:

$$a [Z \to XZ]$$
$$a [X \to XX] \qquad b [X \to \epsilon]$$

$$\text{start } [Z] \to q_0 \qquad q_1 \qquad q_2$$
$$c [Z \to Z]$$
$$c [X \to X] \qquad \epsilon [Z \to \epsilon]$$

Fact 3: DCFL$_{ES}$ ⊂ DCFL

All DCFL$_{ES}$s are DCFLs **BUT** there exists a DCFL that is not a DCFL$_{ES}$.

- $\boxed{\text{DCFL}_{ES} \subseteq \text{DCFL}}$: For a given DCFL$_{ES}$ L and its corresponding DPDA P by **empty stacks**, we can always construct a DPDA P' that accepts L by **final states** as follows:

Fact 3: $DCFL_{ES} \subset DCFL$

All $DCFL_{ES}$s are DCFLs **BUT** there exists a DCFL that is not a $DCFL_{ES}$.

- $\boxed{DCFL \backslash DCFL_{ES} \neq \varnothing}$: The following language is a DCFL but not a $DCFL_{ES}$:

$$L = \{a^n b^n \mid n \geq 0\} \in DCFL \setminus DCFL_{ES}$$

  Why?

Fact 4: DCFL$_{ES}$ = DCFL having Prefix Property

A language L is a DCFL$_{ES}$ if and only if L is a DCFL having the **prefix property**.

Definition (Prefix Property)

A language L has the **prefix property** if and only if for any word $w \in L$, any proper prefix of w is not in L:

$$\forall x, y \in \Sigma^*. \left((xy \in L \land y \neq \epsilon) \Longrightarrow x \notin L \right)$$

For example, the following language is a **DCFL** but does **NOT** have the **prefix property**:

$$L = \{a^n b^n \mid n \geq 0\}$$

because $\epsilon \in L$ is a proper prefix of $ab \in L$.
Thus, L is a **DCFL** but **NOT** a **DCFL$_{ES}$**.

Fact 5: RL $\not\subset$ DCFL$_{ES}$

Fact 5: RL $\not\subset$ DCFL$_{ES}$

There exists a RL that is not a DCFL$_{ES}$.

- $\boxed{\text{RL} \setminus \text{DCFL}_{ES} \neq \varnothing}$: For example, the following language is a **RL** but does **NOT** have the **prefix property**:

$$L = \{a^n \mid n \geq 0\}$$

because aa $\in L$ is a proper prefix of aaaa $\in L$.
Thus, L is a **RL** but **NOT** a **DCFL$_{ES}$**.

$$L \in \text{RL} \setminus \text{DCFL}_{ES}$$

Definition (Inherent Ambiguity)

A language L is **inherently ambiguous** if all CFGs whose languages are L are ambiguous. (i.e. there is no unambiguous grammar for L)

- **Is there any DCFL that is inherently ambiguous?**
  (i.e., is there any DCFL always having ambiguous grammars?)
- **Is there any DCFL that is not inherently ambiguous?**
  (i.e., is there any DCFL having at least one unambiguous grammar?)
- **Is there any inherently ambiguous language that is not a DCFL?**
  (i.e., is there any unambiguous grammar whose language is not a DCFL?)

Fact 6: DCFL $\subset$ Non Inherently Ambiguous Languages

> ### Fact 6: DCFL $\subset$ Non Inherently Ambiguous Languages
> All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

- A DCFL$_{ES}$ has an unambiguous grammar : For a given DCFL$_{ES}$ L and its corresponding DPDA P, we can define a CFG for P as follows:
  - For all $0 \leq j < n$,
    $$S \rightarrow A_{0,j}^Z$$
  - For all $q_i \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$, consider any $(q_j, X_1 \cdots X_m) \in \delta(q_i, a, X)$ and $0 \leq k_1, \cdots, k_m < n$. Then,
    $$A_{i,k_m}^X \rightarrow a\, A_{j,k_1}^{X_1}\ A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m}$$

  For any word $w \in L$, w has a unique moves from the initial configuration to the final configuration in P. And, we know that:
  $$A_{i,j}^X \Rightarrow^* w \qquad \text{if and only if} \qquad (q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$$

  Thus, the above CFG is **unambiguous**.

Fact 6: DCFL $\subset$ Non Inherently Ambiguous Languages

All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

- A DCFL has an unambiguous grammar : For a given DCFL L, we can define another DCFL L' with a special symbol \$ as follows:

$$L' = L\$ = \{w\$ \mid w \in L\}$$

Then, L' is a DCFL_{ES} because it has the prefix property. Thus, L' has an unambiguous grammar G'. Now, we can define an unambiguous grammar G for L by treating \$ as a variable with the rule $\$ \to \epsilon$.

For example, if the given DCFL is $L = \{a^n b^n \mid n \geq 0\}$, then $L' = \{a^n b^n \$ \mid n \geq 0\}$ is a DCFL_{ES} and its unambiguous grammar G' is:

$$S \to X\$ \qquad X \to \texttt{a}X\texttt{b} \mid \epsilon$$

Then, the unambiguous grammar G for L is:

$$S \to X\$ \qquad X \to \texttt{a}X\texttt{b} \mid \epsilon \qquad \$ \to \epsilon$$

Fact 6: DCFL $\subset$ Non Inherently Ambiguous Languages

All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

- $\boxed{\text{Non Inherently Ambiguous Languages} \backslash \text{DCFL} \neq \varnothing}$: For example, the following language is a non inherently ambiguous language but not a DCFL:

$$L = \{ww^R \mid w \in \{a, b\}^*\}$$

because the following unambiguous grammar G represents L:

$$S \rightarrow aSa \mid bSb \mid \epsilon$$

Summary

$\mathcal{P}(\Sigma*)$

CFL

$\mathsf{a}^i\mathsf{b}^j\mathsf{c}^k$
$(i = j \vee j = k)$

DCFL

$\mathsf{a}^n\mathsf{b}^n$

$\mathrm{DCFL_{ES}}$

$\mathsf{a}^n\mathsf{c}\mathsf{b}^n$

ww^R

RL

a^n

$\mathsf{a}^n\mathsf{b}$

Non Inherently
Ambiguous Languages
$\left(\begin{array}{c}\text{At-least-one}\\ \text{Unambiguous Grammars}\end{array}\right)$
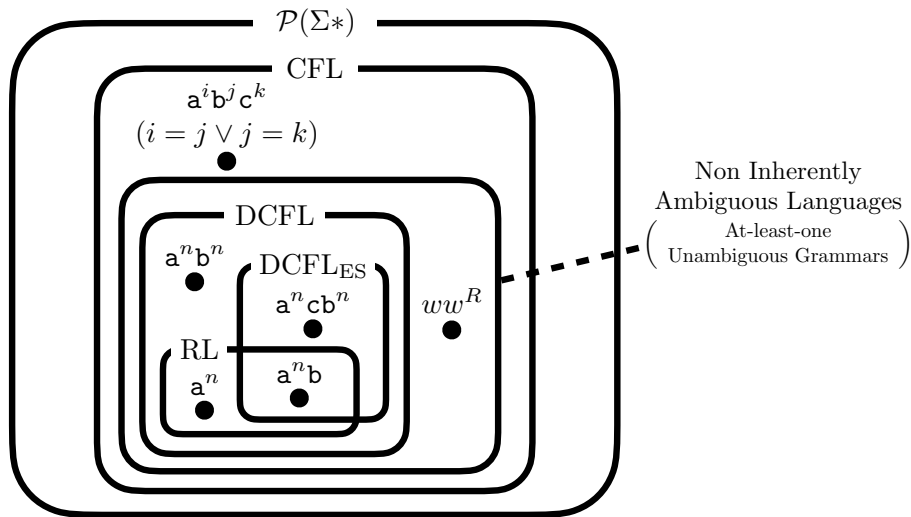
- Normal Forms of Context-Free Grammars

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