Problem 1

A **Fibonacci sequence** is a sequence of integers in which each number is the sum of the two preceding ones. The sequence is like:

$$0, 1, 1, 2, 3, 5, 8, 13, 21 \dots$$

Formally, the Fibonacci sequence is defined as follows.

$$\begin{aligned} &F_0 = 0,\ F_1 = 1 \\ &F_n = F_{n-1} + F_{n-2} \quad (\text{for } n \geq 2) \end{aligned}$$

Problem 1.1

Prove the following theorem.

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

Solution 1.1

(Base case) For $n = 1$, $F_1 = 1 = 2 - 1 = F_3 - 1$.
(Inductive case) Let inductive hypothesis(I.H) holds for $n = k$ $(n \geq 1)$. Then

$$\begin{aligned} \sum_{i=1}^{k+1} F_i &= \sum_{i=1}^{k} + F_{k+1} \\ &= F_{k+2} - 1 + F_{k+1} \\ &= F_{k+3} - 1 \end{aligned}$$

Thus I.H holds for $n = k + 1$. Proved by a induction on integers. □

Problem 1.2

Prove the following theorem.

$$\sum_{i=1}^{n+1} F_i^2 = F_{n+1} F_{n+2}$$

Solution 1.2

(Base case) For $n = 0$, ${F_1}^2 = F_1 F_2 = 1$.

(Inductive case) Let inductive hypothesis(I.H) holds for $n = k$ $(n \geq 0)$. Then

$$\begin{aligned}\sum_{i=1}^{k+2} {F_i}^2 &= \sum_{i=1}^{k+1} {F_i}^2 + {F_{k+2}}^2 \\ &= F_{k+1}F_{k+2} + {F_{k+2}}^2 \\ &= F_{k+2}(F_{k+1} + F_{k+2}) \\ &= F_{k+2}F_{k+3}\end{aligned}$$

Thus I.H holds for $n = k + 1$. Proved by a induction on integers. $\square$

Problem 2

A **parentheses string** p is a string(possibly empty) consisting of opening parenthesis '(' and closing parenthesis ')'. Also, each opening parenthesis should have a proper closing pair. For example, below are the parentheses string:

- ()
- (())()
- ((((()))))

Otherwise, below are not:

- (()
- (()))(

Problem 2.1

Obviously, a set of parentheses strings is the **language**. Give a precise mathematical definition of this language.

Solution 2.1

We write a concatenate of two strings s_1 and s_2 as $s_1 \parallel s_2$.

The language of the parentheses strings L is defined as a smallest set meets conditions below:

(Base case) An empty string $\epsilon \in L$.

(Inductive case 1) If $s \in L$, then '(' $\parallel s \parallel$ ')'$\in L$.

(Inductive case 2) If $s_1 \in L$ and $s_2 \in L$, then $s_1 \parallel s_2 \in L$.

Note: You may define the language using a property of prefixes of the string, but it is burden.

Problem 2.2

Let L be the language defined in Problem 2.1. We define a new language K with

$$s_1 s_2 \dots s_n \in L \iff \overline{s_1 s_2 \dots s_n} \in K$$

when $\overline{s}$ is a conjugate of s; for example, $\overline{\texttt{()(())}} = \texttt{)()))((}$.

Prove or disprove that $L = K^R$.

Solution 2.2

We prove that $L = K^R$.

Proof. We depends on the induction. Also we abuse a notation s^R for reversed s.

(Base case) $\epsilon \in K^R$ is trivial.

(Inductive case 1)

If '(' $\|$ s $\|$ ')' $\in L$, then ')' $\| \overline{s} \|$ '(' $\in K$. Thus '(' $\| \overline{s}^R \|$ ')' $\in K^R$.

(Inductive case 2) If $s_1 \in L$ and $s_2 \in L$, then $\overline{s_1} \;\|\; \overline{s_2} \in K$. Thus $\overline{s_2}^R \;\|\; \overline{s_2}^R \in K^R$.

Thus for $K^R = \{\overline{s}^R \mid s \in L\}$, all conditions of the parentheses string holds. However it is trivial that $|L| = |K^R|$, therefore those are the same. □

Problem 3

A **directed graph** is a pair $G = (V, E)$, where V is a set whose elements are called *vertices*, and E is a set of ordered pairs (x, y) of distinct vertices, formally,

$$E \subseteq \{(x, y) \mid (x, y) \in V \times V \wedge x \neq y\}.$$

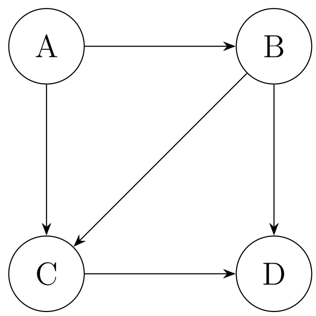

Figure 1: $V = \{A, B, C, D\}$ and $E = \{(A, B), (A, C), (B, D), (B, C), (C, D)\}$.

Hint: You may refer to the inductive definition of the directed graph.

Let (V, E) is a directed graph.

(Base case) $(\varnothing, \varnothing)$ is a directed graph.

(Inductive case 1) For $v \notin V$, $(V \cup \{v\}, E)$ is a directed graph.

(Inductive case 2) For $(x, y) \notin E$, $(V, E \cup \{(x, y)\})$ is a directed graph when $x \neq y$ and $x, y \in V$.

Problem 3.1

The possible number of different directed graphs of n vertices is $2^{n(n-1)}$. Prove it by **induction**(on integers).

Solution 3.1

(Base case) $|\{(\varnothing, \varnothing)\}| = 1$

(Inductive case) Let I.H holds for $n = k$. For a digraph $G = (V, E)$ and $|G'| = k + 1$, let $G' = (V \cup \{v\}, E \cup E')$ while $v \notin V$ and $\forall(x, y) \in E'. (x = v \vee y = v) \wedge x \neq y$. It's trivial that $E \cap E' = \varnothing$ and it is a equivalent condition for valid E'. $|\mathcal{P}(E')| = 2^{2k}$, thus $2^{k(k-1)} \cdot 2^{2k} = 2^{k(k+1)}$.

Thus I.H holds for $n = k + 1$. Proved by a induction on integers. $\square$

Problem 3.2

Let in(x) of the vertex x be the number of edges pointing to it, and out(x) is the number of edges starting from it. For example, in(B) = 1 and out(B) = 2. Prove or disprove the theorem below using a **structural induction** or giving a **counterproof**.

$$\forall G = (V, E).\ \sum_{v \in V} \text{in}(v) = \sum_{v \in V} \text{out}(v)$$

Solution 3.2

(Base case) Trivial for $(\varnothing, \varnothing)$.

(Inductive case 1) For $(V \cup \{v\}, E)$, $\text{in}(v) = \text{out}(v) = 0$. Use I.H.

(Inductive case 2) Consider $(x, y) \notin E$, $G' = (V, E \cup \{(x, y)\})$ when $x \neq y$ and $x, y \in V$. Then $\sum_{v \in V} \text{in}(v)$ of G' increases by 1, since $\sum_{v \in V - \{y\}} \text{in}(v)$ is same and $\text{in}(y)$ increases by 1. In the same way, $\text{out}(x)$ increases by 1.

Now it's proved by a induction. □