- **Submission of this exercise is NOT mandatory**. This exercise will not be reflected in the assessment. Also you are free to seek help from the internet and discuss it with others; even asking for help from a TA is fine.
- If you wish, you can send your answers to TA(kimjg1119@korea.ac.kr) by 23:59 on March 17, 2024, to request grading. Answers submitted after this time are not guaranteed- it depends on whether the teaching assistant is tired.
- Only PDF format is accepted. I **strongly recommend** using LaTeXor markdown with MathJax to write down your answer. If you are unfamiliar with these, the handwritten answer is accepted, but make sure your writing is readable.

Problem 1

A **Fibonacci sequence** is a sequence of integers in which each number is the sum of the two preceding ones. The sequence is like:

$$0, 1, 1, 2, 3, 5, 8, 13, 21 \ldots$$

Formally, the Fibonacci sequence is defined as follows.

$$\begin{aligned} &F_0 = 0,\ F_1 = 1 \\ &F_n = F_{n-1} + F_{n-2} \quad (\text{for } n \geq 2) \end{aligned}$$

Problem 1.1

Prove the following theorem.

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

Problem 1.2

Prove the following theorem.

$$\sum_{i=1}^{n+1} F_i^2 = F_{n+1} F_{n+2}$$

Problem 2

A **parentheses string** p is a string(possibly empty) consisting of opening parenthesis '`(`' and closing parenthesis '`)`'. Also, each opening parenthesis should have a proper closing pair. For example, below are the parentheses string:

- `()`
- `(())()`
- `((()))`

Otherwise, below are not:

- `(()`
- `(()))(`

Problem 2.1

Obviously, a set of parentheses strings is the **language**. Give a precise mathematical definition of this language.

Problem 2.2

Let L be the language defined in Problem 2.1. We define a new language K with

$$s_1 s_2 \ldots s_n \in L \iff \overline{s_1 s_2 \ldots s_n} \in K$$

when $\overline{s}$ is a conjugate of s; for example, $\overline{\texttt{()(())}} = \texttt{)())((}$.
Prove or disprove that $L = K^R$.

Problem 3

A **directed graph** is a pair $G = (V, E)$, where V is a set whose elements are called *vertices*, and E is a set of ordered pairs (x, y) of distinct vertices, formally,

$$E \subseteq \{(x, y) \mid (x, y) \in V \times V \wedge x \neq y\}.$$

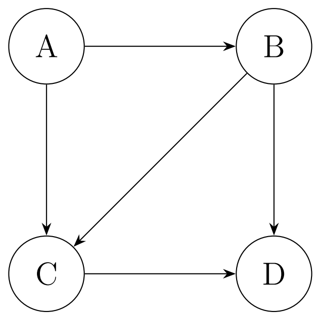

Figure 1: $V = \{A, B, C, D\}$ and $E = \{(A, B), (A, C), (B, D), (B, C), (C, D)\}$.

Hint: You may refer to the inductive definition of the directed graph.

Let (V, E) is a directed graph.

(Base case) $(\varnothing, \varnothing)$ is a directed graph.

(Inductive case 1) For $v \notin V$, $(V \cup \{v\}, E)$ is a directed graph.

(Inductive case 2) For $(x, y) \notin E$, $(V, E \cup \{(x, y)\})$ is a directed graph when $x \neq y$ and $x, y \in V$.

Problem 3.1

The possible number of different directed graphs of n vertices is $2^{n(n-1)}$. Prove it by **induction**(on integers).

Problem 3.2

Let $\text{in}(x)$ of the vertex x be the number of edges pointing to it, and $\text{out}(x)$ is the number of edges starting from it. For example, $\text{in}(B) = 1$ and $\text{out}(B) = 2$. Prove or disprove the theorem below using a **structural induction** or giving a **counterproof**.

$$\forall G = (V, E).\ \sum_{v \in V} \text{in}(v) = \sum_{v \in V} \text{out}(v)$$