Final Exam

COSE215: Theory of Computation

2024 Spring

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- **If you are not good at English, please write your answers in Korean.**
  (영어가 익숙하지 않은 경우, 답안을 한글로 작성해 주세요.)
- **Write answers in good handwriting.**
  If we cannot recognize your answers, you will not get any points.
  (글씨를 알아보기 힘들면 점수를 드릴 수 없습니다. 답안을 읽기 좋게 작성해주세요.)
- **Write your answers in the boxes provided.**
  (답안을 제공된 박스 안에 작성해 주세요.)

Student ID	
Student Name	

Question:	1	2	3	4	5	6	7	8	Total
Points:	15	15	15	10	10	15	10	10	100
Score:									

1. 15 points A **pushdown automaton (PDA)** $P = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$ is a 7-tuple where:

- Q is a finite set of **states**
- Σ is a finite set of **symbols**
- Γ is a finite set of **stack alphabets**
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \to \mathcal{P}(Q \times \Gamma^*)$ is a **transition function**
- $q_0 \in Q$ is the **initial state**
- $Z \in \Gamma$ is the **initial stack alphabet** (the stack is initially Z)
- $F \subseteq Q$ is a set of **final states**

The languages $L_E(P)$ accepted by **empty stacks** of PDA P is defined as:

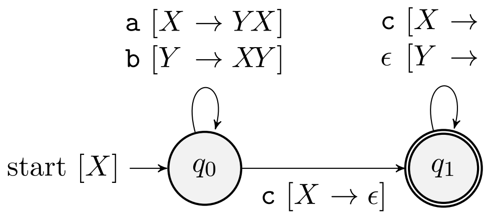
$$L_E(P) = \{w \in \Sigma^* \mid (q_0, w, Z) \vdash^* (q, \epsilon, \epsilon) \text{ for some } q \in Q\}$$

The language $L_F(P)$ accepted by **final states** of a PDA P is defined as:

$$L_F(P) = \{w \in \Sigma^* \mid (q_0, w, Z) \vdash^* (q, \epsilon, \alpha) \text{ for some } q \in F, \alpha \in \Gamma^*\}$$

Fill in the blanks about the following **pushdown automaton (PDA)** P_1.

$P_1 =$



(a) 5 points $L_E(P_1) = \{$ ______ $\}$

(b) 5 points $L_F(P_1) = \{$ ______ $\}$

(c) 5 points The following **context-free grammar (CFG)** represents the language $L_E(P_1)$ accepted by the **empty stacks** of the PDA P_1:

$S \to$ ______

$A^X_{0,1} \to$ ______

$A^X_{1,1} \to$ ______

$A^Y_{0,1} \to$ ______

$A^Y_{1,1} \to$ ______

Note that each variable $A^X_{i,j}$ should generate all words that cause the PDA to move from the state q_i to the state q_j by popping the alphabet X:

$$A^X_{i,j} \Rightarrow^* w \qquad \text{if and only if} \qquad (q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$$

and its right-hand side should consist of terminals and variables $A^X_{0,1}$, $A^X_{1,1}$, $A^Y_{0,1}$, $A^Y_{1,1}$. You can omit the useless (non-generating and unreachable) variables.

2. 15 points A **deterministic pushdown automaton (DPDA)** is a **PDA** that has **at most one** one-step move ($\vdash$) from any configuration. A language is a **deterministic context-free language (DCFL)** if it is accepted by **final states** of some DPDA. A language is a **deterministic context-free language by empty stacks (DCFL_{ES})** if it is accepted by **empty stacks** of some DPDA.

   (a) 8 points Design a **DPDA** P_2 using a **transition diagram** whose language $L_F(P_2)$ accepted by **final states** is equal to the following language L_1:

$$L_F(P_2) = L_1 = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k \geq 1 \land i + j \leq 2k\}$$

$P_2 =$

   (b) 3 points Explain **why** the language L_1 is **not** a $\textbf{DCFL}_{\textbf{ES}}$ with a concrete example.

   (c) 2 points Give an example of a **regular expression** whose language is **not** a $\textbf{DCFL}_{\textbf{ES}}$. If there is no such regular expression, write down **"none"**.

$R =$

   (d) 2 points Give an example of a **regular expression** whose language is a $\textbf{DCFL}_{\textbf{ES}}$. If there is no such regular expression, write down **"none"**.

$R =$

3. 15 points Consider the following **context-free grammar (CFG)** G_0:

$$G_0 = \begin{cases} S \to A \mid \mathtt{a}BB \\ A \to \mathtt{a}C \mid A \\ B \to \epsilon \mid \mathtt{a}B\mathtt{b} \mid C \\ C \to AA \end{cases}$$

(a) 4 points Construct a CFG G_1 consisting of productions produced by replacing **nullable variables** with ϵ in all combinations and removing all ϵ-productions in production rules in G_0:

$G_1 =$

(b) 4 points Construct a CFG G_2 by removing all **unit productions** and adding all possible **non-unit productions** of Y to X for each **unit pair** (Y, X) in G_1:

$G_2 =$

(c) 4 points Construct a CFG G_3 by removing all productions that contain **non-generating variables** or come from **unreachable variables** in G_2:

$G_3 =$

(d) 3 points Construct a CFG G_4 by putting G_3 in **Chomsky normal form (CNF)**:

$G_4 =$

4. [10 points] Design a **PDA** P_3 whose language $L_E(P_3)$ accepted by **empty stacks** of P_3 is equal to the following language L_2 using a **transition diagram**:

$$L_E(P_3) = L_2 = \{w \in \{\texttt{(}, \texttt{)}\}^* \mid (w \text{ is a balanced}) \land N_{\texttt{(}}(w) \equiv 1 (\text{mod } 3)\}$$

where $N_{\texttt{(}}(w)$ denotes the number of `(`'s in w.

$P_3 =$ [ ]

5. [10 points] Fill in the blanks in the **proof** showing that the language L_3 is **not** a **context-free language (CFL)** using the **pumping lemma** for **CFLs**.

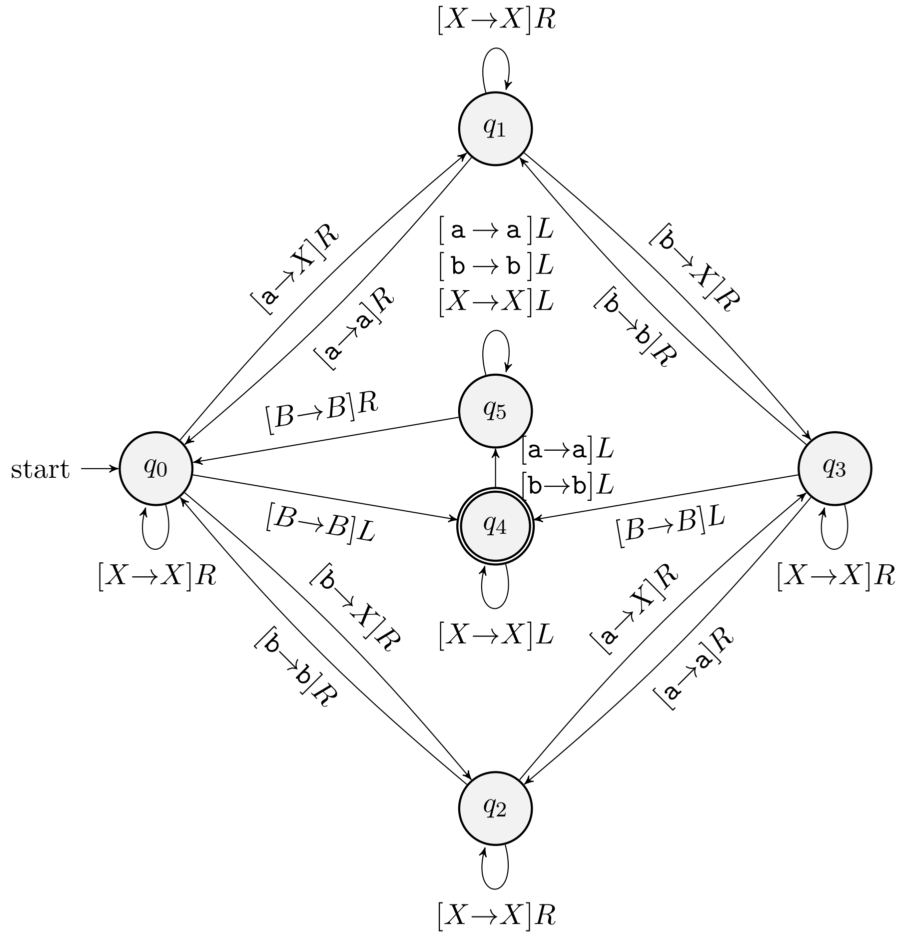
$$L_3 = \{\texttt{a}^i\texttt{b}^j\texttt{c}^i \mid i, j \geq 0 \land i \geq 2j\}$$

1. Assume that any positive integer n is given. (i.e., $n \geq 1$)
2. Pick a word $L_3 \ni z =$ [ ] .
3. $|z| =$ [ ] $\geq n$.
4. Assume that any split $z = uvwxy$ satisfying ① $|vx| > 0$ and ② $|vwx| \leq n$ is given.
5. Let $i =$ [ ] . We need to show that ¬③ $uv^iwx^iy \notin L_3$:

[ ]

6. 15 points Consider the following **Turing machine (TM)** M_1 and fill in the blanks:

$M_1 =$



(a) 8 points The language $L(M_1)$ accepted by the TM M_1 is:

$$L(M_1) = \left\{ w \in \{\texttt{a}, \texttt{b}\}^* \;\middle|\; \boxed{} \right\}$$

(b) 7 points Explain why the **time complexity** of the TM M_1 is $O(n \log_2 n)$.

7. [10 points] Draw a **Turing machine (TM)** M_2 **computing** a function $f : \{\mathtt{a}\}^* \to \{\mathtt{a}\}^*$:

$$f(\mathtt{a}^n) = \mathtt{a}^{2^n} \qquad \text{where} \qquad n \geq 0$$

$M_2 =$

8. [10 points] **True/False questions.** Answer O for True and X for False. (Note that each question is worth **1 points**, but you will get **-1 points** for each wrong answer. So, if you are unsure about the answer, leave it blank.)

1. There are uncountably many recursively enumerable languages (RELs). ☐
2. If a language is defined by an unambiguous grammar, it is DCFL. ☐
3. All NP problems are polynomial-time reducible to a NP-hard problem. ☐
4. A language $L = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\} \setminus \{\mathtt{a}^{100}\mathtt{b}^{100}\}$ is a CFL. ☐
5. All CFLs can be recognized by empty stacks of a PDA having a single state. ☐
6. The universal language $L_u = \{(M, w) \mid w \in L(M)\}$ is decidable. ☐
7. The intersection of two CFLs is always a non-CFL. ☐
8. There is no NP problem can be solved by a TM in polynomial time. ☐
9. We can always construct a DPDA for any regular language. ☐
10. If P = NP, then NP-complete problems are in P. ☐

This is the last page.
I hope that your tests went well!