Lecture 10 – Equivalence and Minimization of Finite Automata

COSE215: Theory of Computation

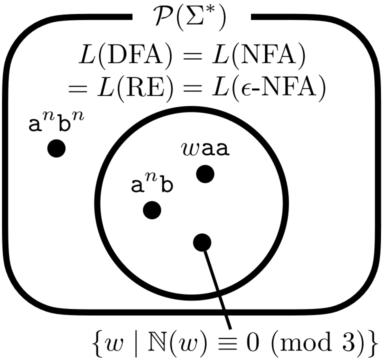
Jihyeok Park

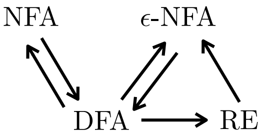
PLRG

2024 Spring

Recall

- Closure Properties of Regular Languages
- Pumping Lemma for Regular Languages





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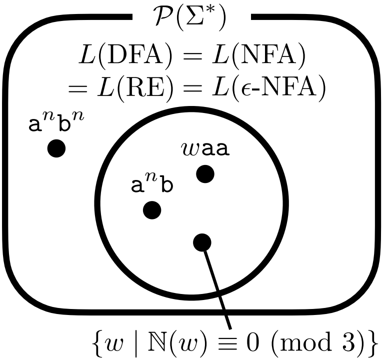

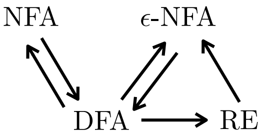

- How to test whether two finite automata are **equivalent**?

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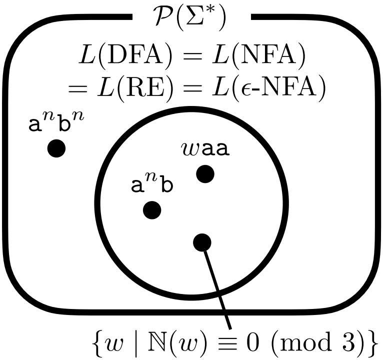

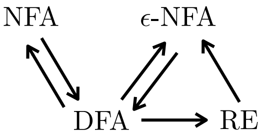

- How to test whether two finite automata are **equivalent**?
- How to **minimize** a finite automaton?

Contents

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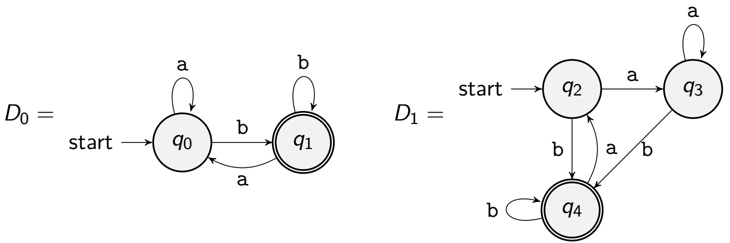
Equivalence of Finite Automata

- Are the following two DFA **equivalent** (i.e., $L(D_0) = L(D_1)$)?



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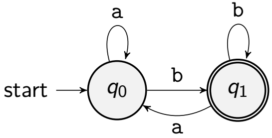

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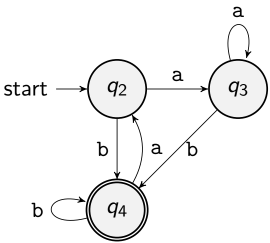

- Yes, because $L(D_0) = L(D_1) = \{w\texttt{b} \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$.

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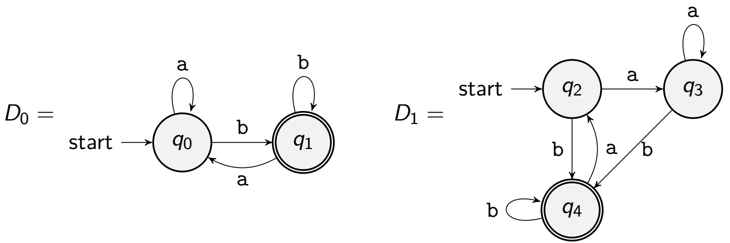

- Yes, because $L(D_0) = L(D_1) = \{w\mathtt{b} \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$.
- We first define the **equivalence of states** and utilize it to test the **equivalence of DFA**.

Equivalence of States ($\equiv$)

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Definition (Equivalence of States ($\equiv$))

For a given DFA D, q_i is **equivalent** to q_j (i.e., $q_i \equiv q_j$) if and only if

$$\forall w \in \Sigma^*.\ \delta^*(q_i, w) \in F \iff \delta^*(q_j, w) \in F$$

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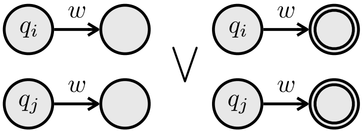

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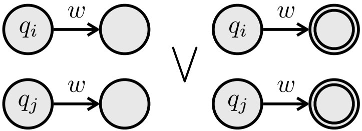

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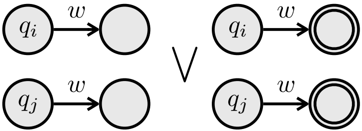

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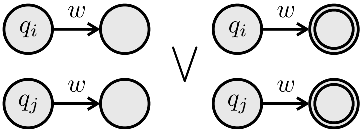

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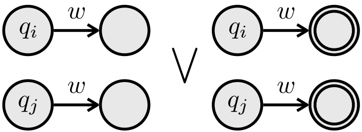

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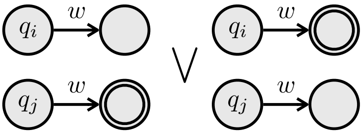

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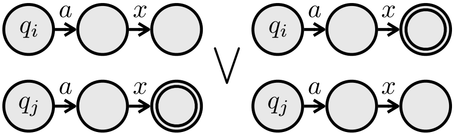

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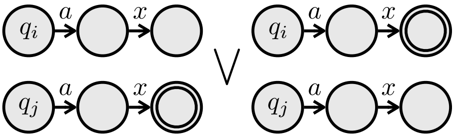

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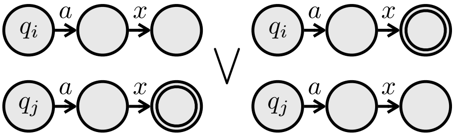

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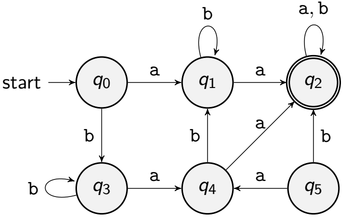

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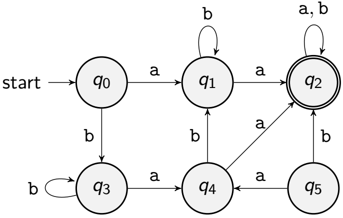

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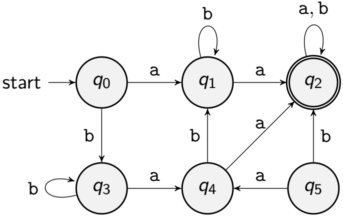

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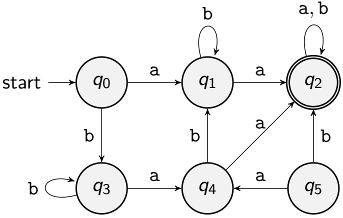

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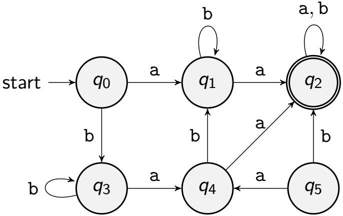

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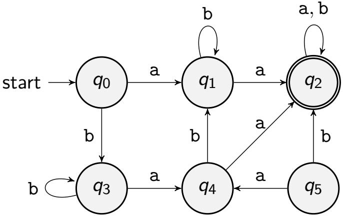

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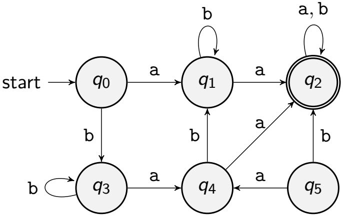
Table-Filling Algorithm



q	a	b
$\rightarrow q_0$	q_1	q_3
q_1	q_2	q_1
$*q_2$	q_2	q_2
q_3	q_4	q_3
q_4	q_2	q_1
q_5	q_4	q_2

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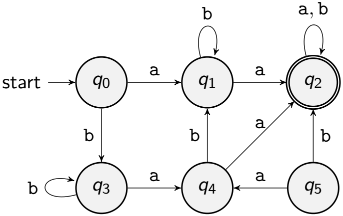

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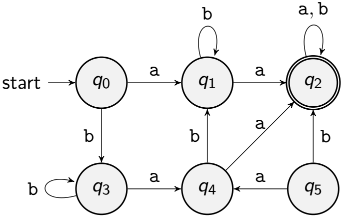

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q_2					
q_3					
q_4					
q_5					

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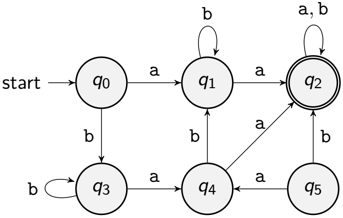

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q_2	x	x			
q_3			x		
q_4			x		
q_5			x		

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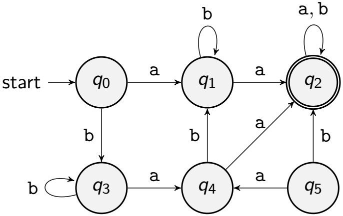

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q_1	X				
q_2	X	X			
q_3		X	X		
q_4	X		X	X	
q_5	X	X	X	X	X
	q_0	q_1	q_2	q_3	q_4

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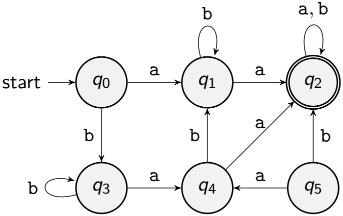

q	a	b
$\rightarrow q_0$	q_1	q_3
q_1	q_2	q_1
$*q_2$	q_2	q_2
q_3	q_4	q_3
q_4	q_2	q_1
q_5	q_4	q_2

(Basis case) $w = \epsilon$.

$$q_i \in F \iff q_j \notin F$$

(Induction case) $w = ax$.

$$\exists a \in \Sigma.\ \delta(q_i, a) \not\equiv \delta(q_j, a)$$

q_1	x				
q_2	x	x			
q_3		x	x		
q_4	x		x	x	
q_5	x	x	x	x	x
	q_0	q_1	q_2	q_3	q_4

$$q_0 \equiv q_3 \wedge q_1 \equiv q_4$$

Equivalence of Finite Automata

Theorem (Equivalence of Finite Automata)

Consider two DFA $D = (Q, \Sigma, \delta, q_0, F)$ *and* $D' = (Q', \Sigma, \delta', q'_0, F')$*. Then,*

$$L(D) = L(D') \iff q_0 \equiv q'_0$$

in a DFA $D'' = (Q \uplus Q', \Sigma, \delta'', q_0, F \uplus F')$ *where*

$$\forall q'' \in Q \uplus Q'. \ \delta''(q, a) = \begin{cases} \delta(q'', a) & q'' \in Q \\ \delta'(q'', a) & q'' \in Q' \end{cases}$$

PLRG

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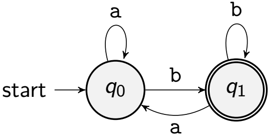
Proof) By the definition of equivalence of states, we have

$$\begin{array}{ll} & L(D) = L(D') \\ \iff & \forall w \in \Sigma^*.\ (D \text{ accepts } w \iff D' \text{ accepts } w) \\ \iff & \forall w \in \Sigma^*.\ (\delta^*(q_0, w) \in F \iff \delta'^*(q_0', w) \in F') \\ \iff & \forall w \in \Sigma^*.\ (\delta''^*(q_0, w) \in F \cup F' \iff \delta''^*(q_0', w) \in F \cup F') \\ \iff & q_0 \equiv q_0' \text{ in } D'' \end{array}$$

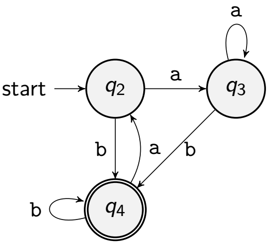
Equivalence of Finite Automata – Example 1

Let's test the equivalence of D_0 and D_1:

$D_0 =$



$D_1 =$



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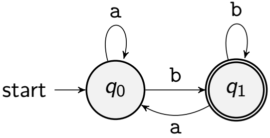

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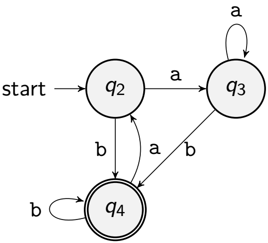

Let's perform the **table-filling algorithm**:

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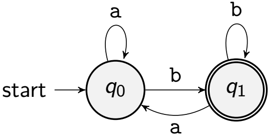

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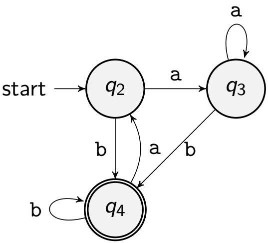

Let's perform the **table-filling algorithm**:

q_1	X			
q_2		X		
q_3		X		
q_4	X		X	X
	q_0	q_1	q_2	q_3

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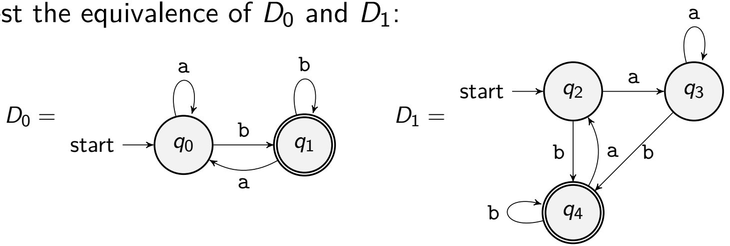

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q_2		x		
q_3		x		
q_4	x		x	x
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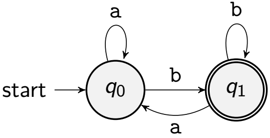

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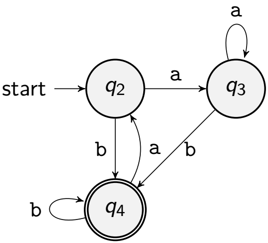

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	q_0	q_1	q_2	q_3

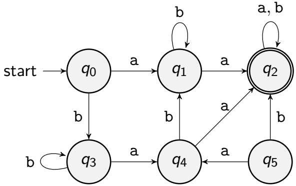
- $q_0 \equiv q_2 \equiv q_3$
- $q_1 \equiv q_4$

$$q_0 \equiv q_2 \implies L(D_0) = L(D_1) = \{w\mathtt{b} \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$$

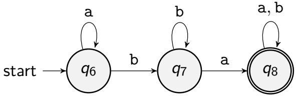
Equivalence of Finite Automata – Example 2

Let's test the equivalence of D_2 and D_3:

$D_2 =$



$D_3 =$



Equivalence of Finite Automata – Example 2

Let's test the equivalence of D_2 and D_3:

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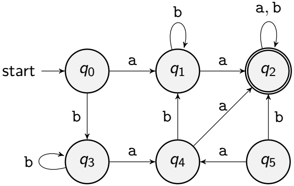

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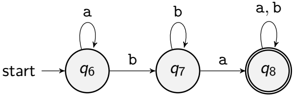

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Equivalence of Finite Automata – Example 2

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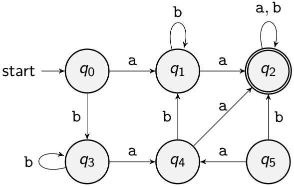

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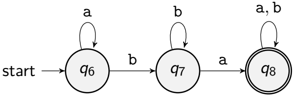
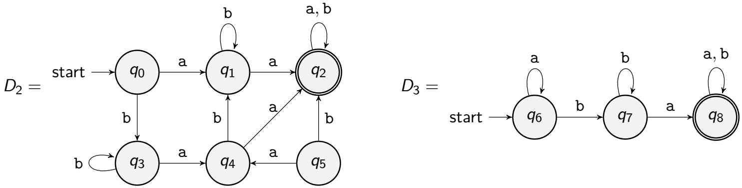


Let's perform the **table-filling algorithm**:

q_1	x							
q_2	x	x						
q_3		x	x					
q_4	x		x	x				
q_5	x	x	x	x	x			
q_6	x	x	x	x	x	x		
q_7	x		x	x		x	x	
q_8	x	x		x	x	x	x	x
	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7

Equivalence of Finite Automata – Example 2

Let's test the equivalence of D_2 and D_3:



Let's perform the **table-filling algorithm**:

q_1	X							
q_2	X	X						
q_3		X	X					
q_4	X		X	X				
q_5	X	X	X	X	X			
q_6	X	X	X	X	X	X		
q_7	X		X	X		X	X	
q_8	X	X		X	X	X	X	X
	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7

- $q_0 \equiv q_3$
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- $q_2 \equiv q_8$
- q_5
- q_6

Equivalence of Finite Automata – Example 2

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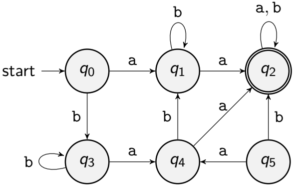

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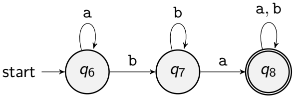


Let's perform the **table-filling algorithm**:

	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
q_1	$\times$							
q_2	$\times$	$\times$						
q_3		$\times$	$\times$					
q_4	$\times$		$\times$	$\times$				
q_5	$\times$	$\times$	$\times$	$\times$	$\times$			
q_6	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$		
q_7	$\times$		$\times$	$\times$		$\times$	$\times$	
q_8	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$	$\times$

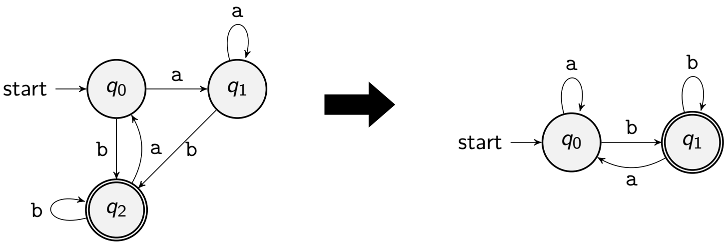
- $q_0 \equiv q_3$
- $q_1 \equiv q_4 \equiv q_7$
- $q_2 \equiv q_8$
- q_5
- q_6

$$q_0 \not\equiv q_6 \implies L(D_2) \neq L(D_3) \;\; (\because \texttt{ba} \notin L(D_2) \text{ but } \texttt{ba} \in L(D_3))$$

Contents

Minimization of Finite Automata

Is it possible to **minimize** a DFA?



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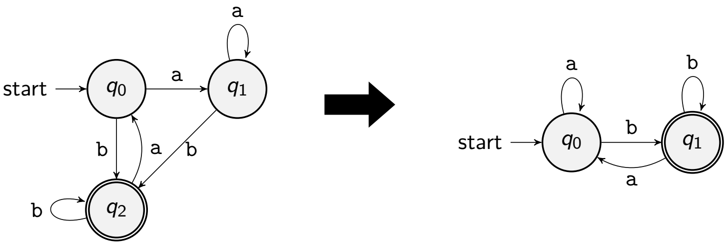

Yes, let's utilize **equivalence classes** $Q/_{\equiv}$ of states defined with $\equiv$.

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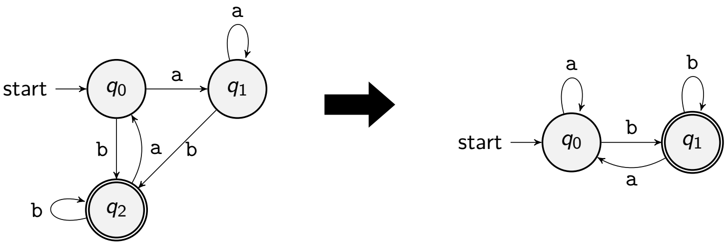

Yes, let's utilize **equivalence classes** $Q/_{\equiv}$ of states defined with $\equiv$.

Note that $\equiv$ is an **equivalence relation**:

- reflexive: $\forall q \in Q.\ q \equiv q$
- symmetric: $\forall q, q' \in Q.\ q \equiv q' \Leftrightarrow q' \equiv q$
- transitive: $\forall q, q', q'' \in Q.\ q \equiv q' \land q' \equiv q'' \Leftrightarrow q \equiv q''$

Minimization Algorithm

For a given DFA $D = (Q, \sigma, \delta, q_0, F)$, the **minimization** algorithm is:

Minimization Algorithm

PLRG

For a given DFA $D = (Q, \sigma, \delta, q_0, F)$, the **minimization** algorithm is:

1. Remove all **unreachable states** from the initial state q_0.

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2. Partition the remaining states into **equivalence classes**:

$$Q/_{\equiv} = \{[q]_{\equiv} \mid q \in Q\}$$

   where the **equivalence class** of a state q is defined as:

$$[q]_{\equiv} = \{q' \in Q \mid q \equiv q'\}$$

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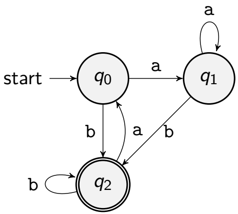
3. Construct a new DFA $D/_{\equiv} = (Q/_{\equiv}, \Sigma, \delta/_{\equiv}, [q_0]_{\equiv}, F/_{\equiv})$ where
   - $\delta/_{\equiv} : Q/_{\equiv} \times \Sigma \to Q/_{\equiv}$ is defined by:

$$\forall q \in Q.\ \forall a \in \Sigma.\ \delta/_{\equiv}([q]_{\equiv}, a) = [\delta(q, a)]_{\equiv}$$

   (We can prove $\forall q', q'' \in [q]_{\equiv}.\ \forall a \in \Sigma.\ [\delta_{\equiv}(q', a)]_{\equiv} = [\delta_{\equiv}(q'', a)]_{\equiv}$.)
   - $F/_{\equiv} = \{[q]_{\equiv} \mid q \in F\}$

Minimization Algorithm - Example 1



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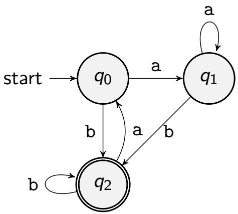
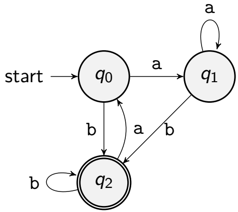


① Remove unreachable states



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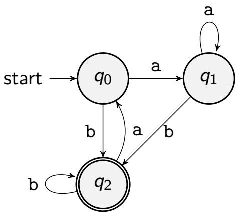

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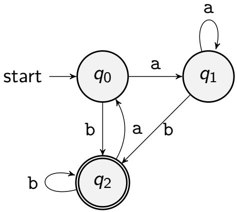
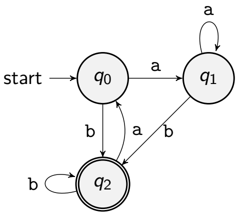


② Partition the states into $Q/_{\equiv}$

$$
\begin{aligned}
Q/_{\equiv} = \{ & \\
& \{q_0, q_1\}, \quad (\because q_0 \equiv q_1) \\
& \{q_2\}, \\
\} &
\end{aligned}
$$

Minimization Algorithm - Example 1



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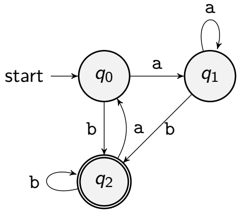

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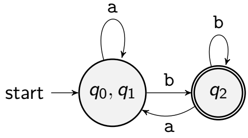
③ Construct a new DFA $D/_{\equiv}$



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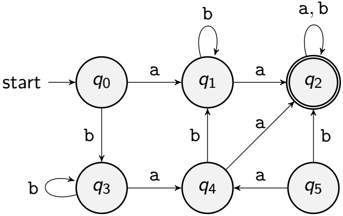

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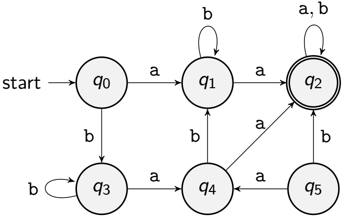

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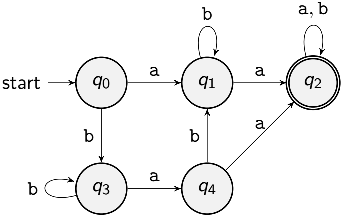

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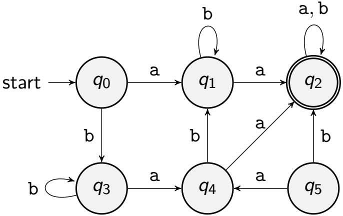

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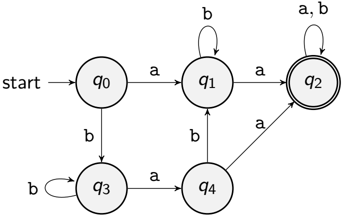

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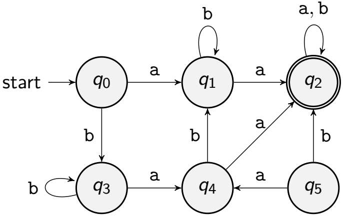

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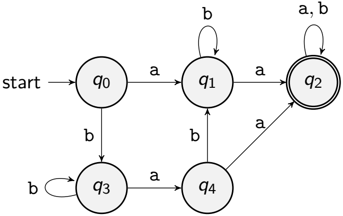

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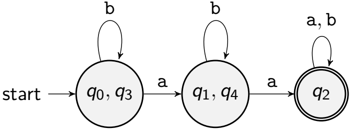

Proof of Minimum-State DFA

Theorem (Minimum-State DFA)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$*, its minimized DFA* $D/_{\equiv} = (Q/_{\equiv}, \Sigma, \delta/_{\equiv}, [q_0]_{\equiv}, F/_{\equiv})$ *is a* **minimum-state DFA** *of* D*.*

(i.e., $\nexists$ *DFA* $D' = (Q', \Sigma, \delta', q_0', F')$*. s.t.* $L(D') = L(D) \land |Q'| < |Q/_{\equiv}|$*).*

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- Assume that $\exists$ DFA D'. Then, $m < n$ when $m = |Q'|$ and $n = |Q/_{\equiv}|$.

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- For any state $q \in Q/_{\equiv}$, we can find a state $q' \in Q'$ such that $q \equiv q'$.

  (We will prove it as a lemma in the next slide.)

Proof of Minimum-State DFA

PLRG

Theorem (Minimum-State DFA)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$*, its minimized DFA* $D/_{\equiv} = (Q/_{\equiv}, \Sigma, \delta/_{\equiv}, [q_0]_{\equiv}, F/_{\equiv})$ *is a* **minimum-state DFA** *of* D.

(i.e., $\nexists$ *DFA* $D' = (Q', \Sigma, \delta', q'_0, F')$*. s.t.* $L(D') = L(D) \wedge |Q'| < |Q/_{\equiv}|$*).*

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  (We will prove it as a lemma in the next slide.)
- By Pigeonhole Principle, $\exists q_i \neq q_j \in Q/_{\equiv}.\ \exists q' \in Q'.\ q_i \equiv q' \wedge q_j \equiv q'$.

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(i.e., $\nexists$ DFA $D' = (Q', \Sigma, \delta', q'_0, F')$. s.t. $L(D') = L(D) \land |Q'| < |Q/_{\equiv}|$).

- Assume that $\exists$ DFA D'. Then, $m < n$ when $m = |Q'|$ and $n = |Q/_{\equiv}|$.
- For any state $q \in Q/_{\equiv}$, we can find a state $q' \in Q'$ such that $q \equiv q'$.

  (We will prove it as a lemma in the next slide.)
- By Pigeonhole Principle, $\exists q_i \neq q_j \in Q/_{\equiv}. \exists q' \in Q'. q_i \equiv q' \land q_j \equiv q'$.
- It means that $q_i \equiv q_j$. However, it contradicts that $Q/_{\equiv}$ is partitioned into equivalence classes of states. □

Proof of Minimum-State DFA – Lemma

Lemma

Consider a given DFA $D = (Q, \Sigma, \delta, q_0, F)$. *Then, let*

- $D/_{\equiv} = (Q/_{\equiv}, \Sigma, \delta/_{\equiv}, [q_0]_{\equiv}, F/_{\equiv})$ *be its minimized DFA*
- $D' = (Q', \Sigma, \delta', q'_0, F')$ *be another DFA such that* $L(D) = L(D')$

Then, for any state $q \in Q/_{\equiv}$, *we can find a state* $q' \in Q'$ *such that* $q \equiv q'$.

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For all $q \in Q/_{\equiv}$. $\exists w = a_1 \cdots a_k$. s.t. $\delta/_{\equiv}(q_0, w) = q$. $\quad (\because q$ is reachable.)

Proof of Minimum-State DFA – Lemma

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Consider a given DFA $D = (Q, \Sigma, \delta, q_0, F)$*. Then, let*

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Let $q' = \delta'(q_0', w)$.

Proof of Minimum-State DFA – Lemma

PLRG

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Let $q' = \delta'(q_0', w)$.

Then, $\delta'^*(q_0', a_1 \cdots a_i) \equiv \delta/_{\equiv}{}^*(q_0, a_1 \cdots a_i)$ for all $0 \leq i \leq k$.

Proof of Minimum-State DFA – Lemma

PLRG

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Consider a given DFA $D = (Q, \Sigma, \delta, q_0, F)$. *Then, let*

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Then, $\delta'^*(q'_0, a_1 \cdots a_i) \equiv \delta/_{\equiv}{}^*(q_0, a_1 \cdots a_i)$ for all $0 \leq i \leq k$.

- **(Basis Case)** $\delta'^*(q'_0, \epsilon) = q'_0 \equiv q_0 = \delta/_{\equiv}{}^*(q_0, \epsilon)$ ($\because$ $L(D') = L(D/_{\equiv})$)

Proof of Minimum-State DFA – Lemma

PLRG

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Consider a given DFA $D = (Q, \Sigma, \delta, q_0, F)$. *Then, let*

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- **(Basis Case)** $\delta'^*(q'_0, \epsilon) = q'_0 \equiv q_0 = \delta/_{\equiv}{}^*(q_0, \epsilon)$ ($\because L(D') = L(D/_{\equiv})$)
- **(Induction Case)** Assume $\delta'^*(q'_0, a_1 \cdots a_i) \not\equiv \delta/_{\equiv}{}^*(q_0, a_1 \cdots a_i)$.

Proof of Minimum-State DFA – Lemma

PLRG

Lemma

Consider a given DFA $D = (Q, \Sigma, \delta, q_0, F)$. *Then, let*

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For all $q \in Q/_{\equiv}$. $\exists w = a_1 \cdots a_k$. s.t. $\delta/_{\equiv}(q_0, w) = q$. $\quad (\because q$ is reachable.)

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- **(Basis Case)** $\delta'^*(q'_0, \epsilon) = q'_0 \equiv q_0 = \delta/_{\equiv}{}^*(q_0, \epsilon) \quad (\because L(D') = L(D/_{\equiv}))$
- **(Induction Case)** Assume $\delta'^*(q'_0, a_1 \cdots a_i) \not\equiv \delta/_{\equiv}{}^*(q_0, a_1 \cdots a_i)$.
  Then, by the definition of distinguishable states,
  $\delta'^*(q'_0, a_1 \cdots a_{i-1}) \not\equiv \delta/_{\equiv}{}^*(q_0, a_1 \cdots a_{i-1})$.

Proof of Minimum-State DFA – Lemma

Lemma

Consider a given DFA $D = (Q, \Sigma, \delta, q_0, F)$. *Then, let*

- $D/_\equiv = (Q/_\equiv, \Sigma, \delta/_\equiv, [q_0]_\equiv, F/_\equiv)$ *be its minimized DFA*
- $D' = (Q', \Sigma, \delta', q'_0, F')$ *be another DFA such that* $L(D) = L(D')$

Then, for any state $q \in Q/_\equiv$, *we can find a state* $q' \in Q'$ *such that* $q \equiv q'$.

For all $q \in Q/_\equiv$. $\exists w = a_1 \cdots a_k$. s.t. $\delta/_\equiv(q_0, w) = q$. $(\because q$ is reachable.)

Let $q' = \delta'(q'_0, w)$.

Then, $\delta'^*(q'_0, a_1 \cdots a_i) \equiv \delta/_\equiv{}^*(q_0, a_1 \cdots a_i)$ for all $0 \leq i \leq k$.

- **(Basis Case)** $\delta'^*(q'_0, \epsilon) = q'_0 \equiv q_0 = \delta/_\equiv{}^*(q_0, \epsilon)$ $(\because L(D') = L(D/_\equiv))$
- **(Induction Case)** Assume $\delta'^*(q'_0, a_1 \cdots a_i) \not\equiv \delta/_\equiv{}^*(q_0, a_1 \cdots a_i)$.
  Then, by the definition of distinguishable states,
  $\delta'^*(q'_0, a_1 \cdots a_{i-1}) \not\equiv \delta/_\equiv{}^*(q_0, a_1 \cdots a_{i-1})$.
  But, it contradicts the induction hypothesis. □

Summary

1. Equivalence of Finite Automata
   - Equivalence of States ($\equiv$)
   - Distinguishable States ($\not\equiv$)
   - Table-Filling Algorithm
   - Equivalence of Finite Automata
   - Examples

2. Minimization of Finite Automata
   - Minimization Algorithm
   - Examples
   - Proof of Minimum-State DFA

Exercise #3

- Please see this document for the exercise.

https://github.com/ku-plrg-classroom/docs/tree/main/cose215/dfa-eq-min

- Please implement the following functions in `Implementation.scala`.
  - `nonEqPairs` for the **table-filling algorithm**.
  - `isEqual` for the **equivalence** of DFAs.
  - `minimize` for the **minimization** of DFAs.
- It is just an exercise, and you **don't need to submit** anything.

Next Lecture

- Context-Free Grammars (CFGs) and Languages (CFLs)

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