Lecture 3 – Deterministic Finite Automata (DFA)

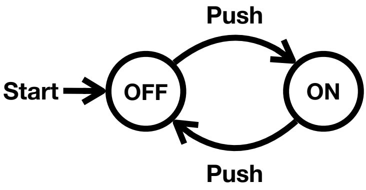
COSE215: Theory of Computation

Jihyeok Park

PLRG

2024 Spring

Recall



1. Mathematical Preliminaries
   - Mathematical Notations
   - Inductive Proofs
   - Notations in Languages
2. Basic Introduction of Scala
   - Basic Features
   - Object-Oriented Programming (OOP)
   - Functional Programming (FP)
   - Immutable Collections (Data Structures)

Contents

Definition of DFA

Definition (Deterministic Finite Automata (DFA))

A **deterministic finite automaton** (DFA) is a 5-tuple:

$$D = (Q, \Sigma, \delta, q_0, F)$$

- Q is a finite set of **states**
- Σ is a finite set of **symbols**
- $\delta : Q \times \Sigma \to Q$ is the **transition function**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

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$$D_1 = (\{q_0, q_1, q_2\}, \{\mathtt{a}, \mathtt{b}\}, \delta, q_0, \{q_2\})$$

$$\begin{array}{lll} \delta(q_0, \mathtt{a}) = q_1 & \delta(q_1, \mathtt{a}) = q_2 & \delta(q_2, \mathtt{a}) = q_2 \\ \delta(q_0, \mathtt{b}) = q_0 & \delta(q_1, \mathtt{b}) = q_0 & \delta(q_2, \mathtt{b}) = q_0 \end{array}$$

Definition of DFA

```scala
// The type definitions of states and symbols
type State = Int
type Symbol = Char
// The definition of DFA
case class DFA(
  states: Set[State],
  symbols: Set[Symbol],
  trans: Map[(State, Symbol), State],
  initState: State,
  finalStates: Set[State],
)
```

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```

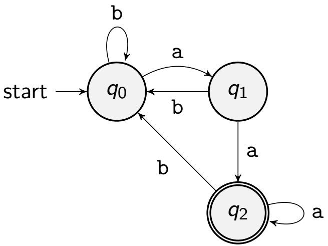
```scala
// An example of DFA
val dfa1: DFA = DFA(
  states      = Set(0, 1, 2),
  symbols     = Set('a', 'b'),
  trans       = Map(
    (0, 'a') -> 1, (1, 'a') -> 2, (2, 'a') -> 2,
    (0, 'b') -> 0, (1, 'b') -> 0, (2, 'b') -> 0,
  ),
  initState   = 0,
  finalStates = Set(2),
)
```

$$D_1 = (\{q_0, q_1, q_2\}, \{\mathtt{a}, \mathtt{b}\}, \delta, q_0, \{q_2\})$$

$$\delta(q_0, \mathtt{a}) = q_1 \qquad \delta(q_1, \mathtt{a}) = q_2 \qquad \delta(q_2, \mathtt{a}) = q_2$$

$$\delta(q_0, \mathtt{b}) = q_0 \qquad \delta(q_1, \mathtt{b}) = q_0 \qquad \delta(q_2, \mathtt{b}) = q_0$$

Transition Diagram



Transition Table

q	a	b
$\rightarrow q_0$	q_1	q_0
q_1	q_2	q_0
$*q_2$	q_2	q_0

Extended Transition Function

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Definition (Extended Transition Function)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **extended transition function** $\delta^* : Q \times \Sigma^* \to Q$ is defined as follows:

- **(Basis Case)** $\delta^*(q, \epsilon) = q$
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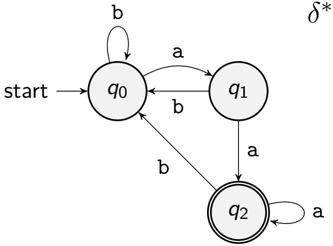
$D_1 =$ 

$\delta^*(q_0, \texttt{baa})$

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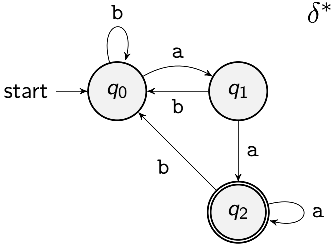

$$\delta^*(q_0, \texttt{baa}) = \delta^*(\delta(q_0, \texttt{b}), \texttt{aa}) = \delta^*(q_0, \texttt{aa})$$

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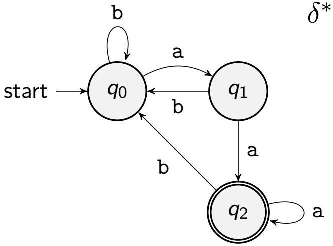
$D_1 =$ 

$$\delta^*(q_0, \texttt{baa}) = \delta^*(\delta(q_0, \texttt{b}), \texttt{aa}) = \delta^*(q_0, \texttt{aa})$$
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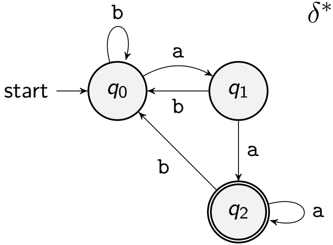

$$
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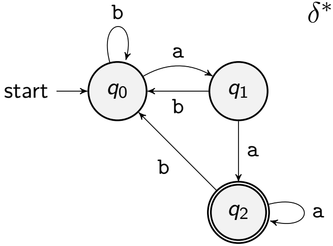

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&= q_2
\end{aligned}
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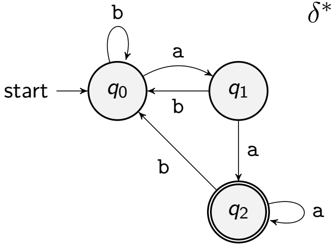
$D_1 =$ 

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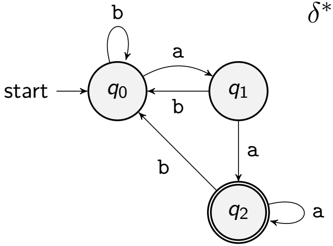

$$\delta^*(q_0, \texttt{aba}) = \delta^*(\delta(q_0, \texttt{a}), \texttt{ba}) = \delta^*(q_1, \texttt{ba})$$

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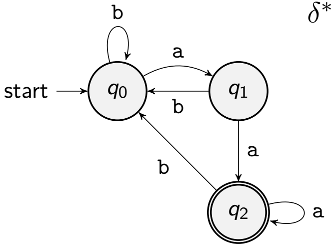

$$\delta^*(q_0, \mathtt{aba}) = \delta^*(\delta(q_0, \mathtt{a}), \mathtt{ba}) = \delta^*(q_1, \mathtt{ba})$$

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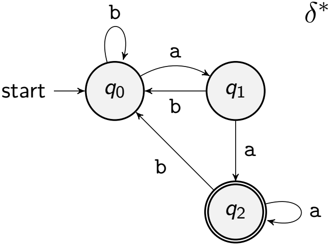
$D_1 =$ 

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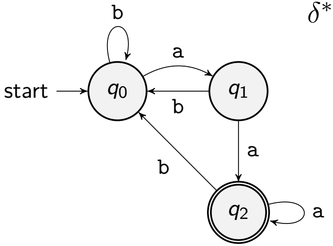
$D_1 =$ 

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&= \delta^*(\delta(q_0, \mathtt{a}), \epsilon) &&= \delta^*(q_1, \epsilon) \\
&= q_1
\end{aligned}
$$

Extended Transition Function

```scala
// The type definition of words
type Word = String
case class DFA(...):
  ...
  // The extended transition function of DFA
  def extTrans(q: State, w: Word): State = w match
    case ""     => q
    case x <| w => extTrans(trans(q, x), w)

// An example transition for a word "baa"
dfa1.extTrans(0, "baa") // 2
// An example transition for a word "aba"
dfa1.extTrans(0, "aba") // 1
```

where <| is a helper function to extract the first symbol and the rest of the word but you do not need to understand the details of how it works.

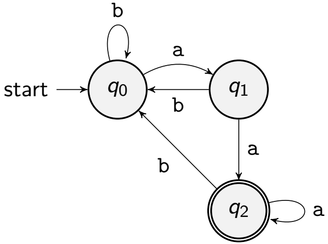
```scala
// A helper function to extract first symbol and rest of word
object `<|` { def unapply(w: Word) = w.headOption.map((_, w.drop(1))) }
```

Acceptance of a Word

Definition (Acceptance of a Word)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, we say that D **accepts** a word $w \in \Sigma^*$ if and only if $\delta^*(q_0, w) \in F$

$D_1 =$



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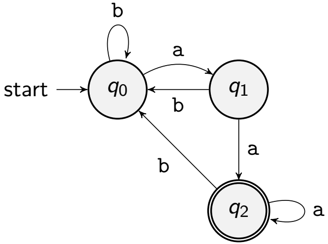

$$\delta^*(q_0, \texttt{baa}) = q_2 \in F$$

It means that D_1 **accepts** `baa`.

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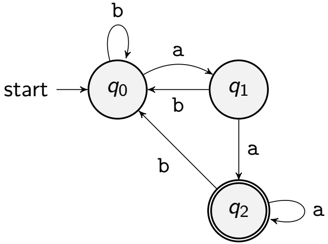
$D_1 =$ 

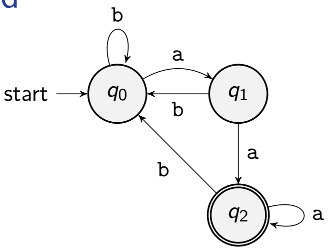
$$\delta^*(q_0, \texttt{baa}) = q_2 \in F$$

It means that D_1 **accepts** `baa`.

$$\delta^*(q_0, \texttt{aba}) = q_1 \notin F$$

It means that D_1 does **not accept** `aba`.

Acceptance of a Word

$D_1 =$ 


```
case class DFA(...):
  ...
  // The acceptance of a word by DFA
  def accept(w: Word): Boolean =
    finalStates.contains(extTrans(initState, w))

// An example acceptance of a word "baa"
dfa1.accept("baa") // true

// An example non-acceptance of a word "aba"
dfa1.accept("aba") // false
```

Language of DFA (Regular Language)

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Definition (Language of DFA)

For a given DFA $D = (Q, \Sigma, \delta, q_0, F)$, the **language** of D is defined as:

$$L(D) = \{w \in \Sigma^* \mid D \text{ accepts } w\}$$

Language of DFA (Regular Language)

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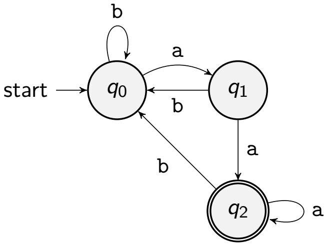
Definition (Regular Language)

A language L is **regular** if and only if there exists a DFA D such that $L(D) = L$

Example 1

$D_1 =$ 

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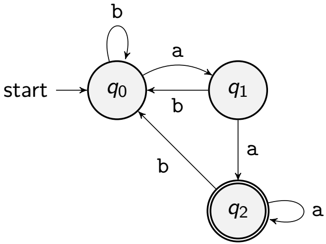

$$\delta^*(q_0, \texttt{baa}) = q_2 \in F$$

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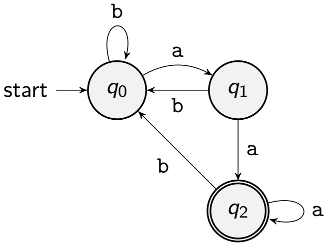
$D_1 =$ 

$\Rightarrow$ $\delta^*(q_0, \texttt{baa}) = q_2 \in F$

D_1 accepts $\texttt{baa}$

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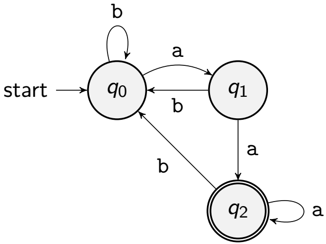

$$
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& \delta^*(q_0, \texttt{baa}) = q_2 \in F \\
\Rightarrow \quad & D_1 \text{ accepts } \texttt{baa} \\
\Rightarrow \quad & \texttt{baa} \in L(D_1)
\end{aligned}
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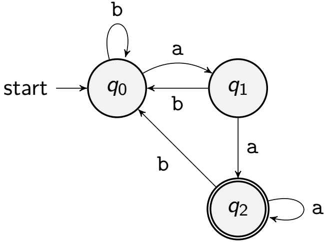
$D_1 =$ 

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$$\epsilon, \texttt{a}, \texttt{b}, \texttt{ab}, \texttt{ba}, \texttt{bb}, \texttt{aab}, \texttt{aba}, \texttt{abb}, \texttt{bab}, \cdots \notin L(D_1)$$

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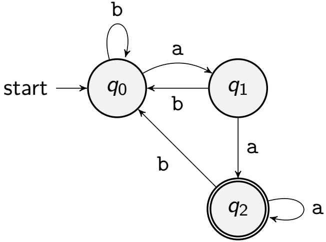
$D_1 =$ 

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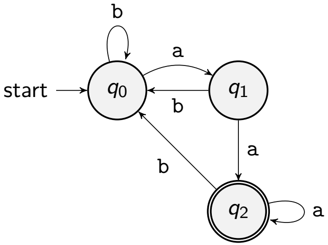

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$$L(D_1) = \{w\texttt{aa} \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$$

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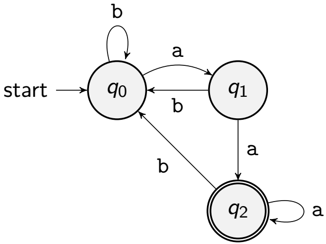

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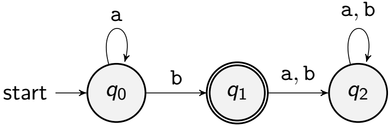
$$L(D_1) = \{w\texttt{aa} \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$$

- q_0 represents ϵ or any word ending with b
- q_1 represents any word ending with exactly one a
- q_2 represents any word ending with at least two a's

Example 2

$D_2 =$



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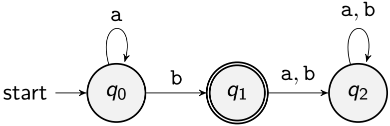
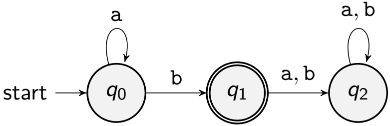


$$\epsilon, \texttt{a}, \texttt{aa}, \texttt{ba}, \texttt{bb}, \texttt{aaa}, \texttt{aba}, \texttt{abb}, \texttt{baa}, \texttt{bab}, \texttt{bba}, \cdots \notin L(D_2)$$

Example 2

$D_2 =$



ϵ, a, aa, ba, bb, aaa, aba, abb, baa, bab, bba, $\cdots \notin L(D_2)$

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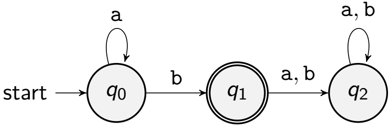

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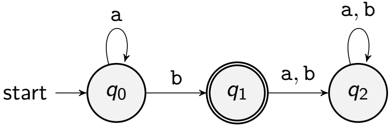

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$$L(D_2) = \{\texttt{a}^n\texttt{b} \mid n \geq 0\}$$

- q_0 represents zero or more a's
- q_1 represents zero or more a's followed by b
- q_2 represents any other words

Example 3

Theorem

The language $L = \{w \in \{0,1\}^* \mid d(w) \equiv 0 \pmod 3\}$ *is regular* ($d(w)$ *is the natural number represented by* w *in binary).*

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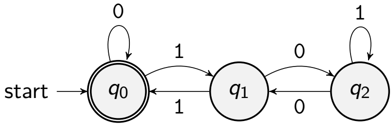

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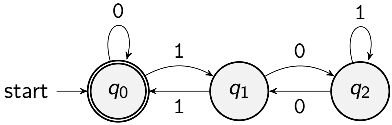

$\square$

- q_0 represents binary format of an integer n s.t. $n \equiv 0 \pmod 3$
- q_1 represents binary format of an integer n s.t. $n \equiv 1 \pmod 3$
- q_2 represents binary format of an integer n s.t. $n \equiv 2 \pmod 3$

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Then, is it possible to prove that L is not regular?

Yes, it is possible BUT you will learn how to prove it (using Pumping Lemma) later in this course.

Summary

1. Deterministic Finite Automata (DFA)
   - Definition
   - Transition Diagram and Transition Table
   - Extended Transition Function
   - Acceptance of a Word
   - Language of DFA (Regular Language)
   - Examples

Next Lecture

- Nondeterministic Finite Automata (NFA)

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