Midterm Exam

COSE215: Theory of Computation

2025 Spring

Instructor: Jihyeok Park

April 23, 2025. 13:30-14:45

- **If you are not good at English, please write your answers in Korean.**
  (영어가 익숙하지 않은 경우, 답안을 한글로 작성해 주세요.)
- **Write answers in good handwriting.**
  If we cannot recognize your answers, you will not get any points.
  (글씨를 알아보기 힘들면 점수를 드릴 수 없습니다. 답안을 읽기 좋게 작성해주세요.)
- **Write your answers in the boxes provided.**
  (답안을 제공된 박스 안에 작성해 주세요.)

Student ID	
Student Name	

Question:	1	2	3	4	5	6	7	8	Total
Points:	15	25	15	5	5	10	10	15	100
Score:									

1. 15 points Design **deterministic finite automata (DFA)** using **transition diagrams** that accept the following languages.

   (a) 5 points $L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid N_{\mathtt{a}}(w) \equiv 1 \pmod 2\}$.

   Note that $N_{\mathtt{a}}(w)$ is the number of a's in w. (e.g., $N_{\mathtt{a}}(\mathtt{baab}) = 2$ and $N_{\mathtt{a}}(\mathtt{abaa}) = 3$.)

   (b) 5 points $L = \{w \in \{\mathtt{0}, \mathtt{1}\}^* \mid$ w does **not** contain 101 as a substring$\}$.

   Note that substrings are continuous sequences of symbols in a word (e.g., 101 is a substring of 010100.)

   (c) 5 points $L \subseteq \{\mathtt{0}, \mathtt{1}\}^*$ such that $h(L) = \{\mathtt{a}^n \mid n \equiv 2 \pmod 5\}$ where $h : \{\mathtt{0}, \mathtt{1}\} \to \{\mathtt{a}\}^*$ is a homomorphism defined as $h(\mathtt{0}) = \mathtt{aa}$ and $h(\mathtt{1}) = \mathtt{aaaa}$.

2. 25 points Write **regular expressions (REs)** that represent the following languages.

(a) 5 points $L = \{w \in \{\texttt{a}, \texttt{b}\}^* \mid w \text{ has at most two } \texttt{a}\text{'s}\}$.

$R =$

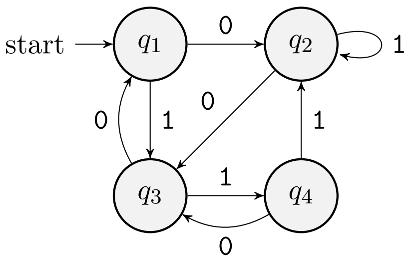
(b) 5 points L is the reversal of the language defined by `(ab|cd*)*ef` (i.e., $L = L(\texttt{(ab|cd}^*\texttt{)}^*\texttt{ef})^R)$.

$R =$

(c) 5 points $L = \{\texttt{a}^n\texttt{b}^m \mid n \times m \equiv 0 \pmod 3\}$.

$R =$

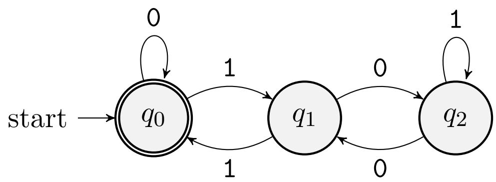
(d) 5 points $L = L(R^{(3)}_{2,4})$ where $R^{(k)}_{i,j}$ be the regular expression that accepts the paths from q_i to q_j whose indices of the intermediate states are bounded by k in the following DFA in the left side:
(Hint: you can use the following regular expressions when $k = 2$ in the right side.)



$R^{(2)}_{1,1} = \epsilon$	$R^{(2)}_{1,2} = \texttt{0|01}^*\texttt{(}\epsilon\texttt{|1)}$	$R^{(2)}_{1,3} = \texttt{1|01}^*\texttt{0}$	$R^{(2)}_{1,4} = \varnothing$
$R^{(2)}_{2,1} = \varnothing$	$R^{(2)}_{2,2} = \texttt{1}^*$	$R^{(2)}_{2,3} = \texttt{1}^*\texttt{0}$	$R^{(2)}_{2,4} = \varnothing$
$R^{(2)}_{3,1} = \texttt{0}$	$R^{(2)}_{3,2} = \texttt{00|001}^*\texttt{(}\epsilon\texttt{|1)}$	$R^{(2)}_{3,3} = \epsilon\texttt{|01|001}^*\texttt{0}$	$R^{(2)}_{3,4} = \texttt{1}$
$R^{(2)}_{4,1} = \varnothing$	$R^{(2)}_{4,2} = \texttt{1|11}^*\texttt{(}\epsilon\texttt{|1)}$	$R^{(2)}_{4,3} = \texttt{0|11}^*\texttt{0}$	$R^{(2)}_{4,4} = \epsilon$

$R^3_{2,4} =$

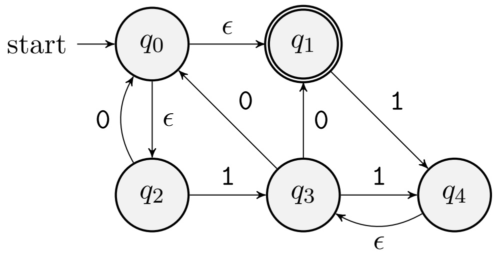
(e) 5 points $L = h(L')$ where $h : \{\texttt{0}, \texttt{1}\} \to \{\texttt{a}, \texttt{b}\}^*$ is a homomorphism defined as $h(\texttt{0}) = \texttt{ab}$ and $h(\texttt{1}) = \texttt{a}$, and $L' \subseteq \{\texttt{0}, \texttt{1}\}^*$ is the language of the following DFA:



$R =$

3. [15 points] Consider the following ϵ-**nondeterministic finite automaton** (ϵ**-NFA**) N^ϵ:

$$N^\epsilon = (Q = \{q_0, q_1, q_2, q_3, q_4\}, \Sigma = \{0, 1\}, \delta, q_0, F = \{q_1\})$$



(a) [6 points] Construct a **DFA** D using a **transition table** such that $L(D) = L(N^\epsilon)$ via the **subset construction**.

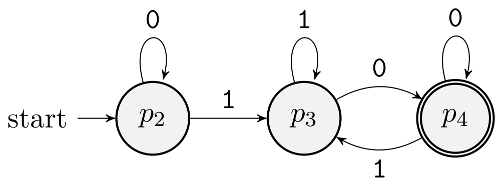
(Note that $\rightarrow$ indicates the **initial state**, and $*$ indicates a **final state**.)

$D =$

	$P = \mathcal{P}(Q)$	0	1
☐	$p_0 = \{$ ☐ $\}$	☐	☐
☐	$p_1 = \{$ ☐ $\}$	☐	☐

(b) [6 points] Consider the following DFA $D' = (P' = \{p_2, p_3, p_4\}, \Sigma, \delta', p_2, \{p_4\})$:

$D' =$



Fill in the table in the left side using the **table-filling algorithm** for the states of D and D', and define the **equivalence classes** $(P \cup P')/_{\equiv}$ in the right side using the result of the algorithm.

p_1	☐			
p_2	☐	☐		
p_3	☐	☐	☐	
p_4	☐	☐	☐	☐
	p_0	p_1	p_2	p_3

$(P \cup P')/_{\equiv} = \{$

$\{$ ☐ $\}$,

$\{$ ☐ $\}$,

$\{$ ☐ $\}$,

$\}$

(c) [3 points] If D and D' are equivalent, **explain why**; otherwise, provide a word $w \in \{0, 1\}^*$ as a **counterexample** that produces different results from D and D'.

4. 5 points Fill in the blanks in the **proofs** to show that $L = \{\mathtt{a}^j \mathtt{b}^k \mid 0 \leq 2j \leq 3k\}$ is **NOT regular**.

   1. Assume that any positive integer n is given. (i.e., $n \geq 1$)
   2. Pick a word $L \ni w =$ ______ .
   3. $|w| =$ ______ $\geq n$.
   4. Assume that any split $w = xyz$ satisfying ① $|y| > 0$ and ② $|xy| \leq n$ is given.
   5. Let $i =$ ______ . We need to show that ¬③ $xy^iz \notin L$:

5. 5 points Consider a DFA $D = (Q, \Sigma, \delta, q_0, F)$ and a word $w \in L(D)$. Let m be the length of w. If $m \geq n$, we can split $w = xyz$ for some $q' \in Q$ satisfying the following conditions:

$$y \neq \epsilon \quad \wedge \quad \delta^*(q_0, x) = q' = \delta^*(q', y) \quad \wedge \quad \delta^*(q', z) \in F$$

For **what value of** n does the above statement always hold? And, **explain why**.

6. [10 points] Design **context-free grammars (CFGs)** that represent the following languages.

   (a) [5 points] $L = \{\texttt{0}^n\texttt{1}^m \mid 0 \leq n \leq m\}$.

   (b) [5 points] $L = \{w \in \{\texttt{a}, \texttt{b}\}^* \mid N_{\texttt{a}}(w) = N_{\texttt{b}}(w) + 1\}$.

   Note that $N_{\texttt{a}}(w)$ and $N_{\texttt{b}}(w)$ are the number of a's and b's in w, respectively.

7. [10 points] Consider the following an **ambiguous CFG** G:

$$S \to A \mid S\texttt{b}S \qquad A \to \texttt{a} \mid \texttt{a}A\texttt{a}$$

   (a) [5 points] Pick a word $w \in L(G)$ that has two different parse trees and draw two parse trees of w:

   $w =$

   tree 1:

   tree 2:

   (b) [5 points] Convert G into an equivalent **unambiguous CFG** with **right-associativity** for b:

8. 15 points A CFG G is called **right-linear** if its all production rules are of the form $A \to x$ or $A \to xB$ where $A, B \in V$ and $x \in \Sigma^*$.

(a) 9 points Prove that the language of any **right-linear** CFG G is **regular** by designing a general algorithm that constructs an **ϵ-NFA** equivalent to the given G.

(b) 6 points Construct an **ϵ-NFA** equivalent to the following **right-linear** CFG using the algorithm you designed in the previous question:

$$S \to \epsilon \mid \texttt{ab}X$$
$$X \to \texttt{a} \mid \texttt{ba}Y$$
$$Y \to S \mid \texttt{cc}X \mid \texttt{d}Y$$

This is the last page.
I hope that your tests went well!