Lecture 1 – Mathematical Preliminaries

COSE215: Theory of Computation

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PLRG

2025 Spring

Roadmap: Towards Turing Machine

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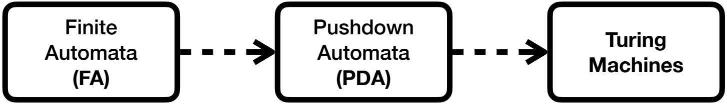
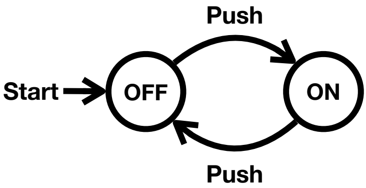
A Turing machine is a specific kind of **automaton**.



- **Part 1: Finite Automata (FA)**
  - Regular Expressions (REs)
  - Regular Languages (RLs)
  - Applications: text search, etc.
- **Part 2: Pushdown Automata (PDA)**
  - Context-Free Grammars (CFGs)
  - Context-Free Languages (CFLs)
  - Applications: programming languages, natural language processing, etc.
- **Part 3: Turing Machines (TMs)**
  - Lambda Calculus (LC)
  - Recursively Enumerable Languages (RELs)
  - Undecidability and Intractability

Recall



Theorem

The current state is OFF *if and only if the button is pushed* **even** *times.*

- Is it possible to prove it?

Let's learn **mathematical background and notation.**

Contents

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Notations in Logics

Notation	Description
A, B	arbitrary **statements**.
$P(x)$	a **predicate** is a statement having variables (e.g., x).
$A \land B$	the **conjunction** of A and B. (i.e., "A and B").
$A \lor B$	the **disjunction** of A and B. (i.e., "A or B").
$\neg A$	the **negation** of A. (i.e., "not A").

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(**Truth Table**) =

A	B	$\neg(A \land B)$	$\neg A \lor \neg B$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

Notations in Logics

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$A \Rightarrow B$	the **implication** of A and B (i.e., "if A then B" or "A implies B") (i.e., $\neg A \vee B$).
$A \Leftrightarrow B$	A **if and only if (iff)** B (i.e., $A \Rightarrow B \wedge B \Rightarrow A$).
$\forall x \in X.\ P(x)$	the **universal quantifier** (i.e., "for all x in X, $P(x)$ holds").
$\exists x \in X.\ P(x)$	the **existential quantifier** (i.e., "there exists x in X such that $P(x)$ holds").

Notations in Set Theory

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$$\textit{where} \qquad a \equiv b \pmod{n} \iff \exists k \in \mathbb{Z}.\ a = b + kn$$

(i.e., a is **congruent** to b **modulo** n)

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$$X \subseteq Y \iff \forall x \in X.\ x \in Y$$

- X is a **proper subset** of Y is denoted by $X \subset Y$.

$$X \subset Y \iff X \subseteq Y \land X \neq Y$$

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- The **difference** of sets

$$X \setminus Y = \{x \mid x \in X \wedge x \notin Y\}$$

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- The **Cartesian product** of sets X and Y is denoted by $X \times Y$.

$$X \times Y = \{(x, y) \mid x \in X \land y \in Y\}$$

Contents

Inductions on Integers

Definition (Inductions on Integers)

Let $P(n)$ be a predicate on integers, and if

- **(Basis Case)** $P(k)$ holds where k is an integer, and
- **(Induction Case)** for all integer $n \geq k$, $P(n) \Rightarrow P(n+1)$,

then $P(i)$ holds for all $i \geq k$.

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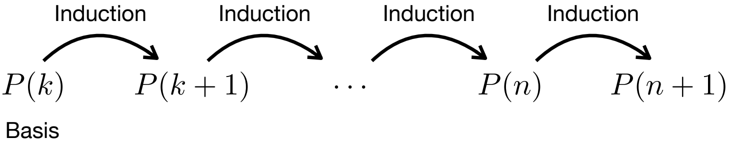

Inductions on Integers – Example 1

Example

Prove that $\forall n \geq 0.\ \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

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$$\begin{aligned} \sum_{i=0}^{n+1} i &= (n+1) + \sum_{i=0}^{n} i \\ &= (n+1) + \frac{n(n+1)}{2} \qquad (\because I.H.) \\ &= \frac{(n+1)(n+2)}{2} \quad \square \end{aligned}$$

Inductions on Integers – Example 2

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Prove that $\forall n \geq 0.\ \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof)

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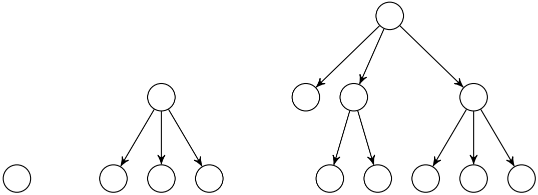
&= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \qquad (\because I.H.) \\
&= \frac{(n+1)(n+2)(2(n+1)+1)}{6} \quad \square
\end{aligned}$$

In CS, we often define somethings as **inductively-defined sets**.
For example, we can define **trees** as follows:

Example (Inductive Definition of Trees)

A **tree** is defined as follows:

- **(Basis Case)** A single **node** N is a tree.
- **(Induction Case)** If $T_1, \cdots, T_n$ are trees, then a graph defined with a new root node N and edges from N to $T_1, \cdots, T_n$ is a tree.



Structural Inductions – Inductive Definitions

Another example is a set of **arithmetic expressions**:

Example (Inductive Definition of Arithmetic Expressions)

An **arithmetic expression** is defined as follows:

- **(Basis Case)** A **number** or a **variable** is an arithmetic expression.
- **(Induction Case)** If E and F are arithmetic expressions, then so are E+F, $E * F$, and (E).

`42`	`x`	`x + y`
`42 * x`	`(x)`	`(x * y) * z`
`(2 + x) * y`	`x * (x * y)`	`((((x))))`

Structural Inductions

Definition (Structural Inductions)

Let $P(x)$ be a predicate on a **inductively-defined set** X, and if

- **(Basis Case)** $P(b_1), \cdots, P(b_k)$ hold for all basis cases $b_1, \cdots, b_k$.
- **(Induction Case)** for all $x \in X$,

$$P(x_1) \land \cdots \land P(x_n) \Rightarrow P(x)$$

  where $x_1, \cdots, x_n$ are the **sub-structures** of x.

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$P(x_1), \cdots, P(x_n)$ are called **induction hypotheses**.

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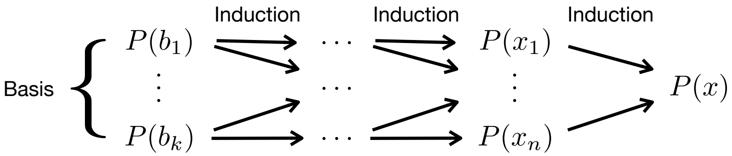

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Prove that for all tree T, the number of nodes in T is equal to the number of edges in T plus one.

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- (Basis Case): $N(T) = 1$ and $E(T) = 0$. □

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- (Basis Case): $N(T) = 1$ and $E(T) = 0$. $\square$
- (Induction Case): Assume that it holds for $T_1, \cdots, T_n$ (I.H.). Then,

$$\begin{aligned} N(T) &= 1 + \sum_{i=1}^{n} N(T_i) \\ &= 1 + \sum_{i=1}^{n} (E(T_i) + 1) \qquad (\because I.H.) \\ &= 1 + n + \sum_{i=1}^{n} E(T_i) \\ &= 1 + E(T) \quad \square \end{aligned}$$

Structural Inductions – Example 2

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Prove that for all arithmetic expression E, the number of left parentheses in E is equal to the number of right parentheses in E.

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- (Induction Case): Assume that it holds for E and F (I.H.). Then,

$$\begin{aligned} L(E{+}F) &= L(E) + L(F) = R(E) + R(F) \qquad (\because I.H.) \\ &= R(E{+}F) \quad \square \end{aligned}$$

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Mutual Inductions

Sometimes, we need to prove multiple predicates simultaneously.

Definition (Mutual Inductions)

Let $P(x)$ and $Q(x)$ are predicates on integers, and if

- **(Basis Case)** $P(k)$ and $Q(k)$ hold where k is an integer, and
- **(Induction Case)** for all $n \geq k$,

$$P(n) \land Q(n) \Rightarrow P(n+1) \land Q(n+1)$$

then $P(i)$ and $Q(i)$ hold for all $i \geq k$.

$P(n)$ and $Q(n)$ are called **induction hypotheses**.

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Sometimes, we need to prove multiple predicates simultaneously.

Definition (Mutual Inductions)

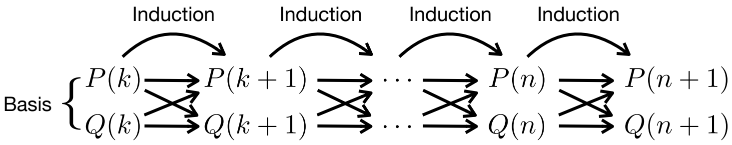
Let $P(x)$ and $Q(x)$ are predicates on integers, and if

- **(Basis Case)** $P(k)$ and $Q(k)$ hold where k is an integer, and
- **(Induction Case)** for all $n \geq k$,

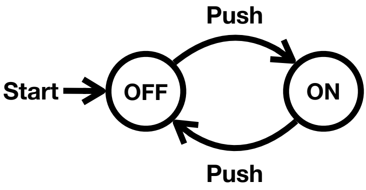
$$P(n) \land Q(n) \Rightarrow P(n+1) \land Q(n+1)$$

then $P(i)$ and $Q(i)$ hold for all $i \geq k$.

$P(n)$ and $Q(n)$ are called **induction hypotheses**.



Mutual Inductions – Example



Theorem

The current state is OFF *if and only if the button is pushed* **even** *times.*

It is difficult to prove it with only one predicate.

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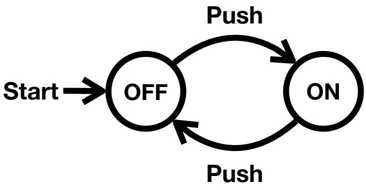

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Let's prove it with two predicates:

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The current state is OFF *if and only if the button is pushed* **even** *times, and the current state is* ON *if and only if the button is pushed* **odd** *times.*

Mutual Inductions – Example

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Mutual Inductions – Example

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Proof) Let $S(i)$ be the current state after i times of pushing. Let's prove

$$\forall i \geq 0.\ S(i) = \text{OFF} \iff i \equiv 0 \pmod 2 \qquad (P)$$

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Mutual Inductions – Example

PLRG

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Mutual Inductions – Example

- (Induction Case): Assume that it holds for n (I.H.):

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PLRG

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$$\epsilon, \text{안녕하세요}, \text{こんにちは}, \bigstar\blacksquare\blacktriangle\oplus, \cdots \in \Sigma^*$$

Symbols & Words

Notation	Description
ϵ	the **empty word**.
$w_1 w_2$	the **concatenation** of w_1 and w_2. (w_1 is a **prefix** of $w_1 w_2$ and w_2 is a **suffix** of $w_1 w_2$)
w^R	the **reverse** of w.
$\lvert w \rvert$	the **length** of w.
Σ^k	the set of all words of length k.
Σ^*	the set of all words (the **Kleene star**). (i.e., $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \cdots = \bigcup_{k \geq 0} \Sigma^k$)
Σ^+	the set of all words except ϵ (the **Kleene plus**). (i.e., $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \cdots = \bigcup_{k \geq 1} \Sigma^k$)

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- $L = \{10, 11, 101, 111, 1011, \cdots\}$ – ???

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- The **union**, **intersection**, and **difference** of languages:

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Languages – Operations

- The **power** of a language defined inductively:

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L^0 &= \{\epsilon\} \\
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Summary

1. Mathematical Notations
   - Notations in Logics
   - Notations in Set Theory

2. Inductive Proofs
   - Inductions on Integers
   - Structural Inductions
   - Mutual Inductions

3. Notations in Languages
   - Symbols & Words
   - Languages

Next Lecture

- Basic Introduction of Scala

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