Lecture 11 – Context-Free Grammars (CFGs) and Languages (CFLs)

COSE215: Theory of Computation

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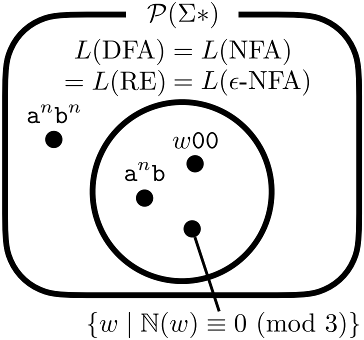
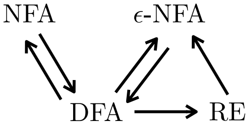
PLRG

2025 Spring

Recall

- Regular Languages
  - Finite Automata - DFA, NFA, ϵ-NFA
  - Regular Expressions





- The minimized DFA is **unique** up to isomorphism by the **Myhill-Nerode Theorem**.[1]

[1] https://en.wikipedia.org/wiki/Myhill-Nerode_theorem

Recall

	Automata	Grammars	Languages
(Part 3) Turing Machines	(Lecture 23) ETM ⇄ (Lecture 21/22) TM ⇄	(Lecture 24) LC	(Lecture 21) REL ∪ DL ⊃ (Lecture 25); (Lecture 26) **NP** $\overset{?}{=}$ **P**, ⊃
(Part 2) Pushdown Automata	(Lecture 14/15) PDA_{FS} ⇄ PDA_{ES} ⇄ (Lecture 16); PDA_{FS} ∪ DPDA_{FS} ⊃ DPDA_{ES} (Lecture 17); DPDA_{FS} ∪, DPDA_{ES} ⋃̸	(Lecture 11/12) CFG; **Chomsky Normal Form** (Lecture 18)	(Lecture 11) CFL; (Lecture 13) **Parse Trees & Ambiguity**; **Closure Properties** (Lecture 19); **Pumping Lemma** (Lecture 20)
(Part 1) Finite Automata	(Lecture 4) NFA ⇄ (Lecture 3) DFA ⇄ (Lecture 5) ϵ-NFA ← (Lecture 7); DFA → RE; **Equivalence & Minimization** (Lecture 10)	(Lecture 6) RE	(Lecture 3) RL; **Closure Properties** (Lecture 8); **Pumping Lemma** (Lecture 9)
(Part 0) Basic Concepts	(Lecture 1) **Mathematical Preliminaries**	(Lecture 2) **Scala**	

Context-Free Grammars (CFGs)

- Consider the following language:

$$L = \{w \in \{\texttt{(}, \texttt{)}\}^* \mid w \text{ is balanced}\}$$

For example, the following words are in (or not in) L:

$$\begin{array}{lcl} L & \ni & \epsilon,\ \texttt{()},\ \texttt{(())},\ \texttt{()()},\ \texttt{(()())},\ \texttt{(()(()))},\ \texttt{((()))},\ \cdots \\ L & \not\ni & \texttt{(},\ \texttt{)},\ \texttt{)(},\ \texttt{(()},\ \texttt{()))},\ \texttt{(()))},\ \texttt{(()((),}\ \cdots \end{array}$$

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- Is this language regular?

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- Is this language regular? **No**, we can prove that this language is **not regular** using the **Pumping Lemma** (Do it yourself!).

- Is there a way to describe this language?

- Yes, let's learn **Context-Free Grammars (CFGs)**!

Contents

Basic Idea

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- **Base Case:** $\epsilon \in L$
- **Inductive Case:** There are two inductive rules:
  - If $w \in L$, then $(w) \in L$
  - If $w_1, w_2 \in L$, then $w_1 w_2 \in L$

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Context-Free Grammars (CFGs) provide a way to describe languages with such **inductive rules** to generate words in the language.

Context-Free Grammars (CFGs)

Definition (Context-Free Grammar (CFG))

A **context-free grammar** is a 4-tuple:

$$G = (V, \Sigma, S, R)$$

where

- V: a finite set of **variables** (nonterminals)
- Σ: a finite set of **symbols** (terminals)
- $S \in V$: the **start variable**
- $R \subseteq V \times (V \cup \Sigma)^*$: a set of **production rules**.

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$$G = (\{S, A, B\}, \{(,)\}, S, R)$$

where R is defined as:

$$\begin{array}{lll} S \to \epsilon & S \to A & S \to B \\ A \to (S) & B \to SS & \end{array}$$

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$$S \rightarrow \epsilon \mid A \mid B \qquad A \rightarrow (S) \qquad B \rightarrow SS$$

We can simplify the notation using the bar ($|$) notation by **combining** multiple production rules for the **same variable**.

Context-Free Grammars (CFGs)

```
// The definition of variables (nonterminals)
type Nt = String
// The type definitions of symbols (terminals)
type Symbol = Char
// The definition of right-hand side of a production rule
case class Rhs(seq: List[Nt | Symbol])
// The definition of context-free grammars
case class CFG(
  nts: Set[Nt],
  symbols: Set[Symbol],
  start: Nt,
  rules: Map[Nt, List[Rhs]],
)
```

Context-Free Grammars (CFGs)

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```

```scala
// An example of CFG
val cfg: CFG = CFG(
  nts = Set("S", "A", "B"), symbols = Set('(', ')'), start = "S",
  rules = Map(
    "S" -> List(Rhs(List()), Rhs(List("A")), Rhs(List("B"))),
    "A" -> List(Rhs(List('(', "S", ')'))),
    "B" -> List(Rhs(List("S", "S")))
  ),
)
```

Derivation Relations

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Definition (Derivation Relation ($\Rightarrow$))

Consider a CFG $G = (V, \Sigma, S, R)$. If a production rule $A \to \gamma \in R$ exists, the **derivation relation** $\Rightarrow \subseteq (V \cup \Sigma)^* \times (V \cup \Sigma)^*$ is defined as:

$$\alpha A \beta \Rightarrow \alpha \gamma \beta$$

for all $\alpha, \beta \in (V \cup \Sigma)^*$. We say that $\alpha A \beta$ **derives** $\alpha \gamma \beta$.

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Definition (Closure of Derivation Relation ($\Rightarrow^*$))

The **closure of derivation relation** $\Rightarrow^*$ is defined as:

- **(Basis Case)** $\forall \alpha \in (V \cup \Sigma)^*. \ \alpha \Rightarrow^* \alpha$
- **(Induction Case)** $\forall \alpha, \beta, \gamma \in (V \cup \Sigma)^*. \ (\alpha \Rightarrow^* \gamma)$ if

$$(\alpha \Rightarrow \beta) \land (\beta \Rightarrow^* \gamma)$$

Derivation Relations

$$G = (\{S, A, B\}, \{\texttt{(}, \texttt{)}\}, S, R)$$

$$S \to \epsilon \mid A \mid B \qquad A \to \texttt{(}S\texttt{)} \qquad B \to SS$$

A derivation for `(())()`:

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A derivation for `(())()`:

$$\begin{aligned} S &\Rightarrow B &&\Rightarrow SS &&\Rightarrow AS &&\Rightarrow (S)S \\ &\Rightarrow (A)S &&\Rightarrow ((S))S &&\Rightarrow (())S &&\Rightarrow (())A \\ &\Rightarrow (())(S) &&\Rightarrow (())() \end{aligned}$$

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A derivation for `(())()`:

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Thus, we can **derive** (or generate/produce) the word `(())()` from S:

$$S \Rightarrow^* (())()$$

Leftmost and Rightmost Derivations

- **Leftmost Derivation** ($\Rightarrow_L$): always derive the *leftmost* variable.
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For example, the **leftmost derivation** for (())():

$$\begin{aligned} S &\Rightarrow_L B &&\Rightarrow_L SS &&\Rightarrow_L AS \\ &\Rightarrow_L (S)S &&\Rightarrow_L (A)S &&\Rightarrow_L ((S))S \\ &\Rightarrow_L (())S &&\Rightarrow_L (())A &&\Rightarrow_L (())(S) &&\Rightarrow_L (())() \end{aligned}$$

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For example, the **leftmost derivation** for `(())()`:

$$\begin{array}{llllllll} S & \Rightarrow_L & B & \Rightarrow_L & SS & \Rightarrow_L & AS & \\ & \Rightarrow_L & \texttt{(}S\texttt{)}S & \Rightarrow_L & \texttt{(}A\texttt{)}S & \Rightarrow_L & \texttt{((}S\texttt{))}S & \\ & \Rightarrow_L & \texttt{(())}S & \Rightarrow_L & \texttt{(())}A & \Rightarrow_L & \texttt{(())(}S\texttt{)} & \Rightarrow_L \quad \texttt{(())()} \end{array}$$

and, the **rightmost derivation** for `(())()`:

$$\begin{array}{llllllll} S & \Rightarrow_R & B & \Rightarrow_R & SS & \Rightarrow_R & SA & \\ & \Rightarrow_R & S\texttt{(}S\texttt{)} & \Rightarrow_R & S\texttt{()} & \Rightarrow_R & A\texttt{()} & \\ & \Rightarrow_R & \texttt{(}S\texttt{)()} & \Rightarrow_R & \texttt{(}A\texttt{)()} & \Rightarrow_R & \texttt{((}S\texttt{))()} & \Rightarrow_R \quad \texttt{(())()} \end{array}$$

Sentential Forms

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Definition (Sentential Form)

For a given CFG $G = (V, \Sigma, S, R)$, a sequence of variables or symbols $\alpha \in (V \cup \Sigma)^*$ is a **sentential form** if and only if $S \Rightarrow^* \alpha$.

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For example, $(A)S$ is a **left-sentential form**:

$$S \Rightarrow_L B \Rightarrow_L SS \Rightarrow_L AS \Rightarrow_L (S)S \Rightarrow_L (A)S$$

and, $S(S)$ is a **right-sentential form**:

$$S \Rightarrow_R B \Rightarrow_R SS \Rightarrow_R SA \Rightarrow_R S(S)$$

Context-Free Languages (CFLs)

Definition (Language of CFG)

For a given CFG $G = (V, \Sigma, S, R)$, the **language** of G is defined as:

$$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

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A language L is **context-free language (CFL)** if and only if there exists a CFG G such that $L(G) = L$.

$$G = (\{S, A, B\}, \{\texttt{(}, \texttt{)}\}, S, R)$$

$$S \rightarrow \epsilon \mid A \mid B \qquad A \rightarrow \texttt{(}S\texttt{)} \qquad B \rightarrow SS$$

Then, $\texttt{(())()} \in L(G)$ because $S \Rightarrow^* \texttt{(())()}$.

Example 1

What is the language of the following CFG?

$$G = (\{S, A, B\}, \{(,)\}, S, R)$$

$$S \to \epsilon \mid A \mid B \qquad A \to (S) \qquad B \to SS$$

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The language of G is:

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The language of G is:

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In addition, it is equivalent to the following CFG:

$$S \to \epsilon \mid (S)S$$

Example 2

Define a CFG whose language is:

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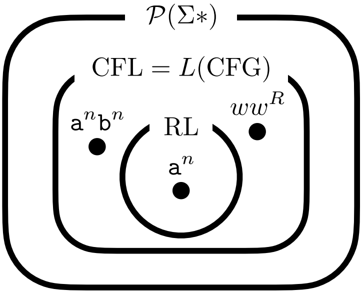
The answer is:

$$S \to \epsilon \mid \mathtt{a}S\mathtt{a} \mid \mathtt{b}S\mathtt{b}$$

Summary

1. Context-Free Grammars (CFGs)

- Definition
- Derivation Relations
- Leftmost and Rightmost Derivations
- Sentential Forms
- Context-Free Languages (CFLs)
- Examples



Next Lecture

- Examples of Context-Free Grammars

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