Lecture 12 – Examples of Context-Free Grammars

COSE215: Theory of Computation

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2025 Spring

Recall

- A **context-free grammar (CFG)**:

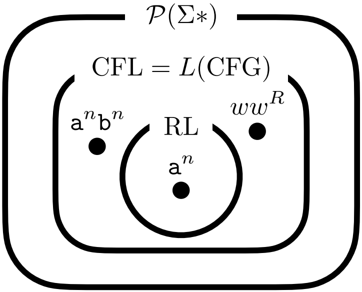
$$G = (V, \Sigma, S, R)$$

- The **language** of a CFG G:

$$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

- A language L is a **context-free language (CFL)**:

$$\exists \text{ CFG } G.\ L(G) = L$$



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Regular Languages are Context-Free

Theorem (RLs are CFLs)

All regular languages are context-free.

Proof) There are two ways to prove this theorem:

1. Converting regular expressions to equivalent CFGs
2. Converting ϵ-NFAs to equivalent CFGs

Regular Expressions to CFGs

For a given regular language L, let's construct an equivalent CFG G using the equivalent regular expression R. $L(G) = L(R)$.

RE R	CFG G
$\varnothing$	$S \to S$
ϵ	$S \to \epsilon$
$a \in \Sigma$	$S \to a$
$R_1 \mid R_2$	$S \to S_1 \mid S_2$
$R_1 \cdot R_2$	$S \to S_1 S_2$
R_1^*	$S \to \epsilon \mid S_1 S$
(R_1)	$S \to S_1$

where S_1 and S_2 are start variables of CFGs G_1 and G_2 such that $L(G_1) = L(R_1)$ and $L(G_2) = L(R_2)$, respectively.

Regular Expressions to CFGs – Examples

For a given RE R, construct a CFG G such that $L(G) = L(R)$.

- $R = \epsilon | \texttt{ab} | \texttt{ba}$

$$S \rightarrow F \mid D \qquad A \rightarrow \texttt{a} \qquad C \rightarrow AB \qquad E \rightarrow \epsilon$$
$$B \rightarrow \texttt{b} \qquad D \rightarrow BA \qquad F \rightarrow E \mid C$$

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Its simplified version:

$$S \to \epsilon \mid \texttt{ab} \mid \texttt{ba}$$

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$$S \to \epsilon \mid AS \qquad A \to \epsilon \mid \mathtt{a}$$

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$$S \rightarrow \epsilon \mid \texttt{ab} \mid \texttt{ba}$$

- $R = (\epsilon|\texttt{a})^*$

$$S \rightarrow \epsilon \mid AS \qquad A \rightarrow \epsilon \mid \texttt{a}$$

- $R = (\texttt{0}|\texttt{1}(\texttt{0}\texttt{1}^*\texttt{0})^*\texttt{1})^*$

$$S \rightarrow \epsilon \mid AS \qquad \begin{array}{l} A \rightarrow \texttt{0} \mid \texttt{1}B\texttt{1} \\ B \rightarrow \epsilon \mid CB \end{array} \qquad \begin{array}{l} C \rightarrow \texttt{0}D\texttt{0} \\ D \rightarrow \epsilon \mid \texttt{1}D \end{array}$$

ϵ-NFAs to CFGs

For a given ϵ-NFA $N^\epsilon = (Q, \Sigma, \delta, q_0, F)$, let's construct a CFG G as:

- For each state $q \in Q$ of N^ϵ, introduce a non-terminal A_q.
- For each transition $q \xrightarrow{a} q'$ of N^ϵ, introduce a production rule:

$$A_q \rightarrow aA_{q'}$$

- For each ϵ-transition $q \xrightarrow{\epsilon} q'$ of N^ϵ, introduce a production rule:

$$A_q \rightarrow A_{q'}$$

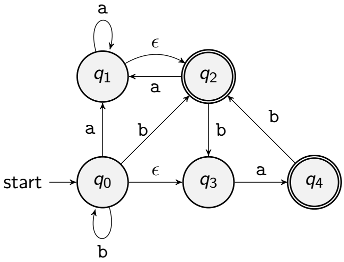
- For each final state $q \in F$ of N^ϵ, introduce a production rule:

$$A_q \rightarrow \epsilon$$

- The start variable of G is A_{q_0}.

ϵ-NFAs to CFGs – Examples

$N^\epsilon =$



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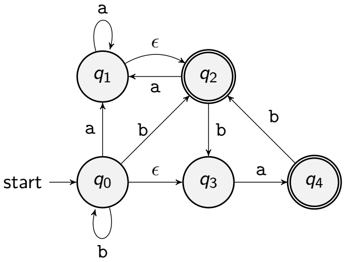

We can construct a CFG G (A_0 is the start variable) for N^ϵ:

$$
\begin{aligned}
A_0 &\to \mathtt{b}A_0 \mid \mathtt{a}A_1 \mid \mathtt{b}A_2 \mid A_3 \\
A_1 &\to \mathtt{a}A_1 \mid A_2 \\
A_2 &\to \mathtt{a}A_1 \mid \mathtt{b}A_3 \mid \epsilon \\
A_3 &\to \mathtt{a}A_4 \\
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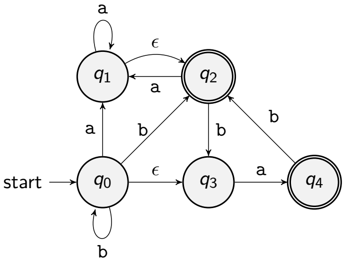

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\end{aligned}$$

For example, we can derive $\mathtt{ba} \in L(N^\epsilon)$ using G:

$$A_0 \Rightarrow \mathtt{b}A_0 \Rightarrow \mathtt{b}A_3 \Rightarrow \mathtt{ba}A_4 \Rightarrow \mathtt{ba}$$

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Example 1: $\mathtt{b}^n\mathtt{a}^m\mathtt{b}^{2n}$

Construct a CFG for the language:

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Therefore, the following is a CFG for L:

$$\begin{array}{l} S \to A \mid \mathtt{b}S\mathtt{bb} \\ A \to \epsilon \mid \mathtt{a}A \end{array}$$

Example 2: Well-Formed Brackets

Construct a CFG for the language:

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Example 3: Equal Number of a's and b's

Construct a CFG for the language:

$$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid N_{\mathtt{a}}(w) = N_{\mathtt{b}}(w)\}$$

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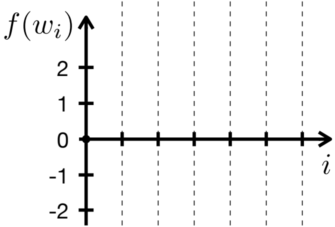

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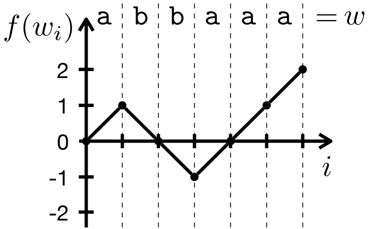

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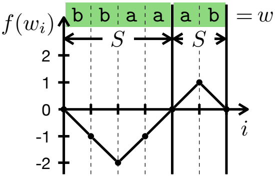

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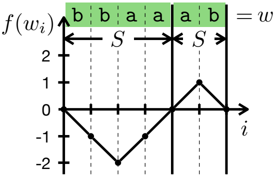

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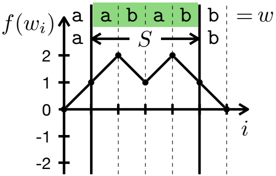

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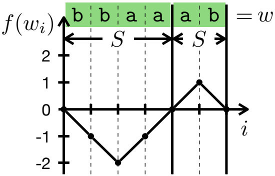

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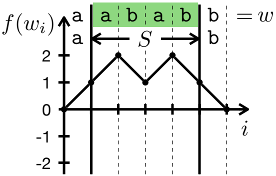

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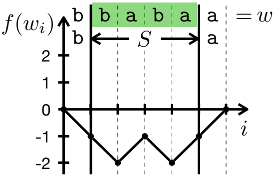

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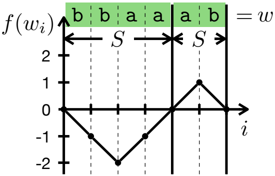

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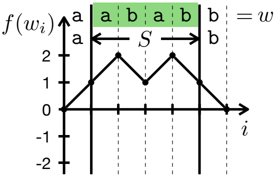

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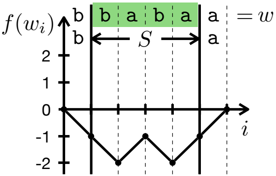

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Therefore, the following is a CFG for L:

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Example 4: Unequal Number of a's and b's

Construct a CFG for the **complement** of the language in Example 3:

$$L = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid N_{\mathtt{a}}(w) \neq N_{\mathtt{b}}(w)\}$$

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We can categorize $w \in \{\texttt{a}, \texttt{b}\}^*$ into three cases using the function f:

- $L_Z = \{w \in \{\texttt{a}, \texttt{b}\}^* \mid f(w) = 0\}$ – equal number of a's and b's
- $L_P = \{w \in \{\texttt{a}, \texttt{b}\}^* \mid f(w) > 0\}$ – more a's than b's
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Let's define production rules for L_P and L_N using graphs for $f(w_i)$.

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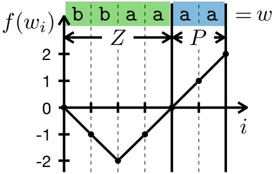

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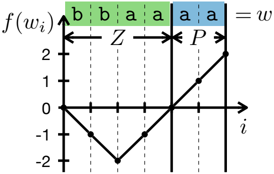

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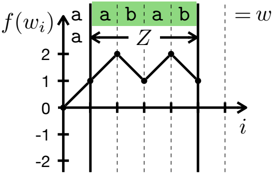


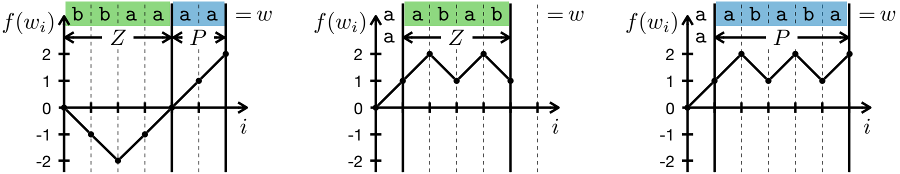
② $w = \mathtt{a}w'$

$(w' \in L_Z)$

Example 4: Unequal Number of a's and b's

Let's split a word $w \in L_P$ using shorter words in L_Z, L_P, and L_N.

For a given $w \in L_P$, there are three cases:



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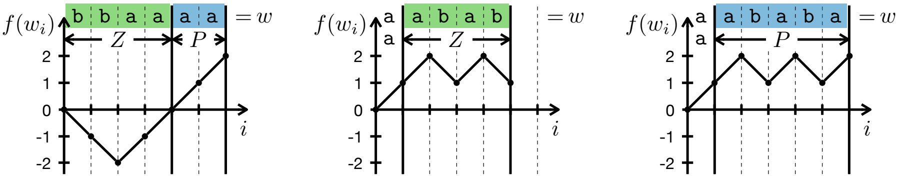

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Therefore, the following is production rules for L_P:

$$P \to ZP \mid \mathtt{a}P \mid \mathtt{a}Z$$

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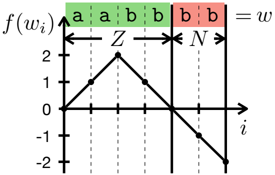

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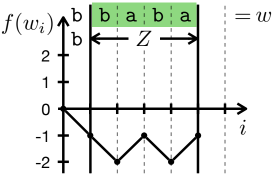
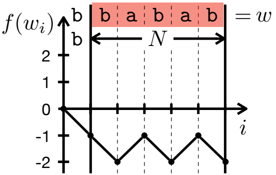


② $w = \mathtt{b}w'$

$(w' \in L_Z)$



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Similarly, the following is production rules for L_N:

$$N \rightarrow ZN \mid \mathtt{b}N \mid \mathtt{b}Z$$

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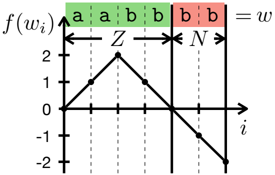

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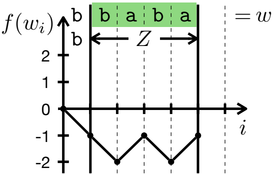
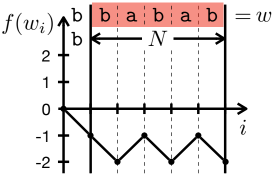


② $w = \mathtt{b}w'$
$(w' \in L_Z)$



③ $w = \mathtt{b}w'$
$(w' \in L_N)$

Similarly, the following is production rules for L_N:

$$N \to ZN \mid \mathtt{b}N \mid \mathtt{b}Z$$

Therefore, the CFG for L is:

$$\begin{aligned} S &\to P \mid N \\ P &\to ZP \mid \mathtt{a}P \mid \mathtt{a}Z \\ N &\to ZN \mid \mathtt{b}N \mid \mathtt{b}Z \\ Z &\to \epsilon \mid ZZ \mid \mathtt{a}Z\mathtt{b} \mid \mathtt{b}Z\mathtt{a} \end{aligned}$$

Example 5: Arithmetic Expressions

An **arithmetic expression** is defined with the following CFG:

$$
\begin{aligned}
S &\rightarrow N \mid X \mid S\texttt{+}S \mid S\texttt{*}S \mid \texttt{(}S\texttt{)} \\
N &\rightarrow D \mid DN \\
D &\rightarrow \texttt{0} \mid \cdots \mid \texttt{9} \\
X &\rightarrow \texttt{a} \mid \cdots \mid \texttt{z}
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\end{aligned}$$

We can **derive** an arithmetic expression `13*(2+x)` as follows:

$$\begin{array}{llllllll}
S & \Rightarrow & S\texttt{*}S & \Rightarrow & N\texttt{*}S & \Rightarrow & DN\texttt{*}S & \Rightarrow & \texttt{1}N\texttt{*}S \\
 & \Rightarrow & \texttt{1}D\texttt{*}S & \Rightarrow & \texttt{13*}S & \Rightarrow & \texttt{13*(}S\texttt{)} & \Rightarrow & \texttt{13*(}S\texttt{+}S\texttt{)} \\
 & \Rightarrow & \texttt{13*(}N\texttt{+}S\texttt{)} & \Rightarrow & \texttt{13*(}D\texttt{+}S\texttt{)} & \Rightarrow & \texttt{13*(2+}S\texttt{)} & \Rightarrow & \texttt{13*(2+}X\texttt{)} \\
 & \Rightarrow & \texttt{13*(2+x)} & & & & & &
\end{array}$$

Example 6: Regular Expressions

Consider a language representing the **syntax of regular expressions**:

$$L = \{w \in \{\varnothing, \varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{|}, \mathtt{*}, \mathtt{(}, \mathtt{)}\}^* \mid w \text{ is a regular expression over } \{\mathtt{a}, \mathtt{b}\}\ \}$$

Example 6: Regular Expressions

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Is this language L **regular**? or **context-free**?

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We can prove that L is **not regular** using the pumping lemma.
(Hint: consider a word $\mathtt{(}^n \varepsilon \mathtt{)}^n$ for a given $n > 0$)

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We can **derive** a regular expression `(b|ab)*` as follows:

$$\begin{array}{llllll} S & \Rightarrow & S\mathtt{*} & \Rightarrow & \mathtt{(}S\mathtt{)*} & \Rightarrow & \mathtt{(}S\mathtt{|}S\mathtt{)*} \\ & \Rightarrow & \mathtt{(}S\mathtt{|}SS\mathtt{)*} & \Rightarrow & \mathtt{(}S\mathtt{|}S\mathtt{b)*} & \Rightarrow & \mathtt{(}S\mathtt{|ab)*} \\ & \Rightarrow & \mathtt{(b|ab)*} \end{array}$$

Example 7: Simplified Scala Syntax

We can define a CFG for a simplified version of Scala syntax[1]:

$$
\begin{array}{lrl}
\text{(Scala Program)} & S \rightarrow & E \mid S \; \texttt{;} \; E \\
\text{(Expressions)} & E \rightarrow & N \mid X \mid E \;\texttt{+}\; E \mid E \;\texttt{-}\; E \mid E \;\texttt{*}\; E \mid E \;\texttt{/}\; E \\
 & \mid & \texttt{val}\; X \texttt{:}\; T \;\texttt{=}\; E \\
 & \mid & \texttt{def}\; X \texttt{(}\; P \;\texttt{):}\; T \;\texttt{=}\; E \\
 & \mid & E \;\texttt{(}\; A \;\texttt{)} \\
 & \mid & \texttt{if}\;\texttt{(}\; E \;\texttt{)}\; E \;\texttt{else}\; E \\
 & \mid & \texttt{enum}\; T \;\texttt{\{}\; D \;\texttt{\}} \\
 & \mid & E \;\texttt{match}\;\texttt{\{}\; C \;\texttt{\}} \\
\text{(Numbers)} & N \rightarrow & \texttt{0} \mid \cdots \mid \texttt{9} \mid \texttt{0}N \mid \cdots \mid \texttt{9}N \\
\text{(Variables)} & X \rightarrow & Y \mid YX \\
 & Y \rightarrow & \varnothing \mid \texttt{a} \mid \cdots \mid \texttt{z} \mid \texttt{A} \mid \cdots \mid \texttt{Z} \\
\text{(Types)} & T \rightarrow & X \mid T \;\texttt{[}\; T \;\texttt{]} \mid T \;\texttt{=>}\; T \\
\text{(Parameters)} & P \rightarrow & \epsilon \mid X \;\texttt{:}\; T \mid P \;\texttt{,}\; X \;\texttt{:}\; T \\
\text{(Arguments)} & A \rightarrow & \epsilon \mid E \mid A \;\texttt{,}\; E \\
\text{(Cases)} & C \rightarrow & \texttt{case}\; E \;\texttt{=>}\; E \mid C \;\texttt{;}\; \texttt{case}\; E \;\texttt{=>}\; E \\
\text{(Enum Cases)} & D \rightarrow & \texttt{case}\; T \;\texttt{(}\; P \;\texttt{)} \mid D \;\texttt{;}\; \texttt{case}\; T \;\texttt{(}\; P \;\texttt{)}
\end{array}
$$

[1] https://docs.scala-lang.org/scala3/reference/syntax.html

Example 7: Simplified Scala Syntax

```
def sum(n: Int): Int = n match { case 0 => 0; case n => n + sum(n - 1) }
```

A derivation for this program:

$S \Rightarrow^* \texttt{def}\ X(\ P\ )\texttt{:}\ T = E \quad \Rightarrow^* \texttt{def sum(}\ P\ \texttt{):}\ T = E$

$\Rightarrow^* \texttt{def sum(}\ X\texttt{:}\ T\ \texttt{):}\ T = E \quad \Rightarrow^* \texttt{def sum(n: Int): Int} = E$

$\Rightarrow^* \texttt{def sum(n: Int): Int} = E\ \texttt{match}\ \{\ C\ \}$

$\Rightarrow^* \texttt{def sum(n: Int): Int = n match}\ \{\ C\ \}$

$\Rightarrow^* \texttt{def sum(n: Int): Int = n match}\ \{\ \texttt{case}\ E \texttt{ => } E\ ;\ C\ \}$

$\Rightarrow^* \texttt{def sum(n: Int): Int = n match \{ case 0 => 0;}\ C\ \}$

$\Rightarrow^* \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; case}\ E \texttt{ => } E\ \}$

$\Rightarrow^* \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; case n =>}\ E\ \}$

$\Rightarrow^* \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; case n =>}\ E + E\ \}$

$\Rightarrow^* \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; case n => n +}\ E\ \}$

$\Rightarrow^* \texttt{def sum(n: Int): Int = n match \{ case 0 => 0; case n => n + sum(n - 1) \}}$

Summary

1. Regular Languages are Context-Free
    - Regular Expressions to CFGs
    - ϵ-NFA to CFG

2. Examples of Context-Free Grammars
    - Example 1: $\mathtt{b}^n\mathtt{a}^m\mathtt{b}^{2n}$
    - Example 2: Well-Formed Brackets
    - Example 3: Equal Number of a's and b's
    - Example 4: Unequal Number of a's and b's
    - Example 5: Arithmetic Expressions
    - Example 6: Regular Expressions
    - Example 7: Simplified Scala Syntax

Midterm Exam

- The midterm exam will be given in class.
- **Date:** 13:30-14:45 (1 hour 15 minutes), April 23 (Wed.).
- **Location:** 301, Aegineung (애기능생활관 301호)
- **Coverage:** Lectures 1 – 13
- **Format:** 7–9 questions with closed book and closed notes
  - Filling blanks in some tables, sentences, or expressions.
  - Construction of automata or grammars for given languages.
  - Proofs of given statements related to languages and automata.
  - Yes/No questions about concepts in the theory of computation.
  - etc.
- Note that there is **no class** on **April 28 (Mon.)**.
- Please refer to the **previous exams** in the course website:

  https://plrg.korea.ac.kr/courses/cose215/

Next Lecture

- Parse Trees and Ambiguity

Jihyeok Park
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