Lecture 13 – Parse Trees and Ambiguity

COSE215: Theory of Computation

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PLRG

2025 Spring

Recall

- A **context-free grammar (CFG)**:

$$G = (V, \Sigma, S, R)$$

- The **language** of a CFG G:

$$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

- A language L is a **context-free language (CFL)**:

$$\exists \text{ CFG } G.\ L(G) = L$$

- For a given word $w \in L(G)$, a **derivation** for w is $S \Rightarrow^* w$
- A sequence $\alpha \in (V \cup \Sigma)^*$ is a **sentential form** if $S \Rightarrow^* \alpha$.

Contents

1. Parse Trees
    - Definition
    - Yields
    - Relationship between Parse Trees and Derivations

2. Ambiguity
    - Ambiguous Grammars
    - Eliminating Ambiguity
    - Inherent Ambiguity

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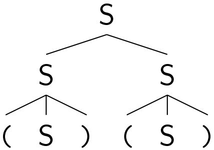
However, **parse trees** focus on the structure of the derivations instead of considering the order of the derivation steps.

For example, the above two derivations have the same parse tree:



Parse Trees

Definition (Parse Trees)

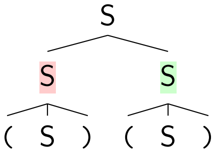
For a given CFG $G = (V, \Sigma, S, R)$, **parse trees** are trees satisfying:

1. The **root node** is labeled with the **start variable** S.
2. Each **internal node** is labeled with a **variable** $A \in V$.
   If its children are labeled with:
   $$X_1, X_2, \cdots, X_k$$
   from the left to the right, then $A \to X_1 X_2 \cdots X_k \in R$.
3. Each **leaf node** is labeled with a variable, symbol, or ϵ. However, if a leaf node is labeled with ϵ, it must be the only child of its parent.

Parse Trees – Example 1: Balanced Parentheses

$$S \rightarrow \epsilon \mid (S) \mid SS$$

A parse tree for $(S)(S)$:



① $S \Rightarrow_L S\,S \Rightarrow_L (S)S \Rightarrow (S)(S)$

② $S \Rightarrow_R S\,S \Rightarrow_R S(S) \Rightarrow (S)(S)$

Parse Trees – Example 2: Even Palindromes

$$S \to \epsilon \mid \mathtt{a}S\mathtt{a} \mid \mathtt{b}S\mathtt{b}$$

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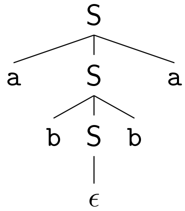

Parse Trees – Example 3: Arithmetic Expressions

$$
\begin{aligned}
S &\rightarrow N \mid X \mid S\texttt{+}S \mid S\texttt{*}S \mid \texttt{(}S\texttt{)} \\
N &\rightarrow D \mid DN \\
D &\rightarrow \texttt{0} \mid \cdots \mid \texttt{9} \\
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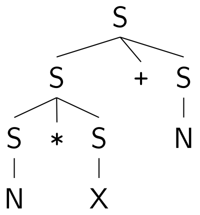
\end{aligned}
$$

A parse tree for $N\texttt{*}X\texttt{+}N$:



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The sequence obtained by concatenating the labels (without ϵ) of the leaf nodes of a parse tree from left to right is called the **yield** of the parse tree.

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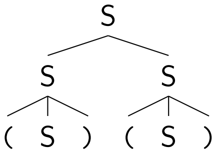

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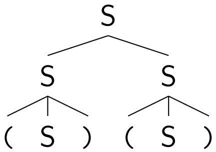

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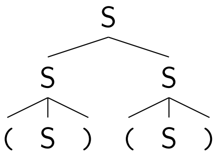

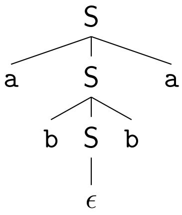

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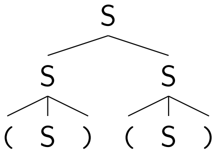

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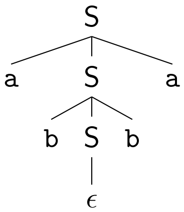

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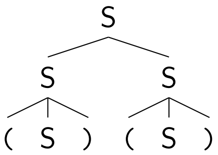

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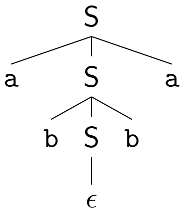

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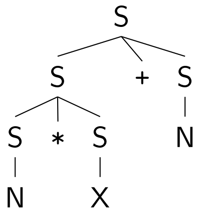

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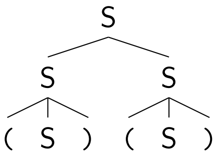

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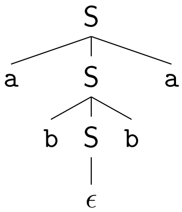

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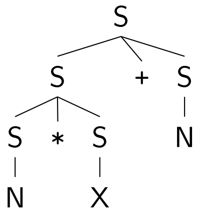

Its yield is $N{*}X{+}N$.

Relationship between Parse Trees and Derivations

Theorem (Parse Trees and Derivations)

For a given CFG $G = (V, \Sigma, S, R)$, *for any sequence* $\alpha \in (V \cup \Sigma)^*$:

$$S \Rightarrow^* \alpha \iff \exists \textit{ parse tree } T.\ \textit{s.t. } T \textit{ yields } \alpha$$

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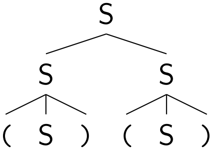
$$\Updownarrow$$



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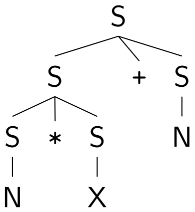

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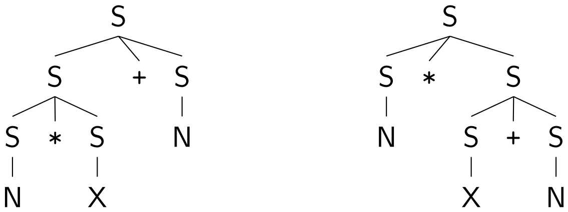

Actually, there are **two** parse trees for $N\texttt{*}X\texttt{+}N$.

Ambiguous Grammars

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A context-free grammar $G = (V, \Sigma, S, R)$ is **ambiguous** if there exist two distinct parse trees for a word $w \in \Sigma^*$. If not, G is **unambiguous**.

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1. *The number of parse trees whose yields are w.*
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Proof) We can convert a left-most (or right-most) derivation for a word w into the corresponding parse tree for w and vice versa.

Ambiguous Grammars – Example

$$
\begin{aligned}
S &\to N \mid X \mid S\texttt{+}S \mid S\texttt{*}S \mid \texttt{(}S\texttt{)} \\
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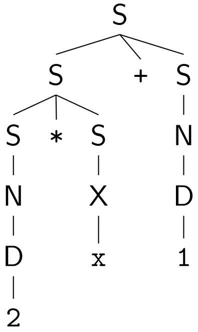
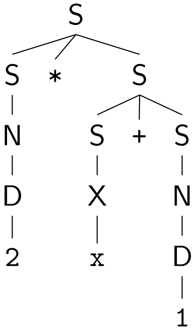
This grammar is **ambiguous** because there are **two** parse trees for the word 2 * x + 1:





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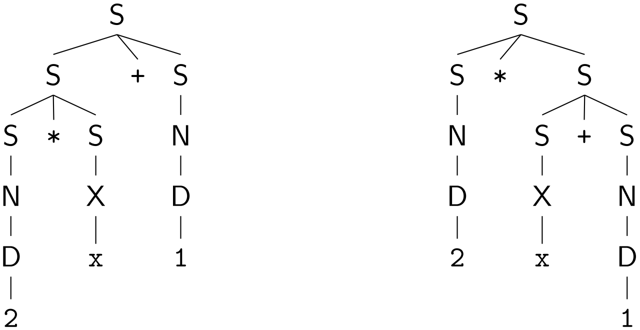

So, there are **two** left-most (or right-most) derivations for `2 * x + 1`.

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There are **two** left-most derivations for `2 * x + 1`:

1. Applying the production rule $S \rightarrow S\texttt{+}S$ first:

$$\begin{aligned} S &\Rightarrow_L S\texttt{+}S \Rightarrow_L S\texttt{*}S\texttt{+}S \Rightarrow_L N\texttt{*}S\texttt{+}S \Rightarrow_L D\texttt{*}S\texttt{+}S \Rightarrow_L \texttt{2*}S\texttt{+}S \\ &\Rightarrow_L \texttt{2*}X\texttt{+}S \Rightarrow_L \texttt{2*x+}S \Rightarrow_L \texttt{2*x+}N \Rightarrow_L \texttt{2*x+}D \Rightarrow_L \texttt{2*x+1} \end{aligned}$$

2. Applying the production rule $S \rightarrow S\texttt{*}S$ first:

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\end{array}
$$

For example, an equivalent but unambiguous grammar is:

$$
\begin{array}{rcl}
S & \rightarrow & T \mid S\texttt{+}T \\
T & \rightarrow & F \mid T\texttt{*}F \\
F & \rightarrow & N \mid X \mid \texttt{(}S\texttt{)} \\
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Now, the unique parse tree for `2 * x + 1` is:

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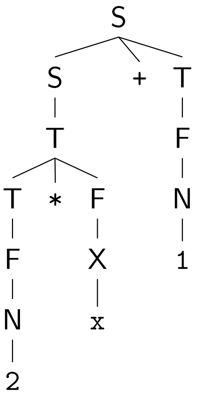

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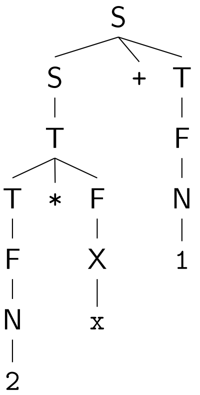

Let's try to understand how to eliminate the ambiguity in the original grammar.

Eliminating Ambiguity

First, analyze why the original grammar is ambiguous.

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- **Precedence** is not specified between different operators (+ and *).
  - For example, two parse trees for `1 * 2 + 3` interpreted as:

    `1 * (2 + 3)` and `(1 * 2) + 3`

  - Let's give * higher precedence than + to interpret it as `(1 * 2) + 3`.

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- **Precedence** is not specified between different operators (`+` and `*`).
  - For example, two parse trees for `1 * 2 + 3` interpreted as:

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  - Let's give `*` higher precedence than `+` to interpret it as `(1 * 2) + 3`.
- **Associativity** for the same operator (`+` or `*`).
  - For example, two parse trees for `1 + 2 + 3` interpreted as:

    `1 + (2 + 3)` and `(1 + 2) + 3`

  - Let's give the left-associativity to `+` to interpret it as `(1 + 2) + 3`.

Eliminating Ambiguity – Precedence

To enforce the **precedence**, define new variables F for factors and T for terms:

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To enforce the **precedence**, define new variables F for factors and T for terms:

- A **factor** is a number, a variable, or a parenthesized expression:

  $$\texttt{42}, \quad \texttt{x}, \quad \texttt{(1 + 2)}, \quad \cdots$$

  In the grammar, F is defined as:

  $$F \to N \mid X \mid (S)$$

Eliminating Ambiguity – Precedence

To enforce the **precedence**, define new variables F for factors and T for terms:

- A **factor** is a number, a variable, or a parenthesized expression:

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- A **term** is the multiplication of one or more factors:

  $$\texttt{42}, \quad \texttt{2 * x}, \quad \texttt{2 * (1 + 2)}, \quad \texttt{1 * (x * y) * z}, \quad \cdots$$

  In the grammar, T is defined as:

  $$T \rightarrow F \mid T * F$$

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- An **expression** is the addition of one or more terms:

  `42,   1 + 2,   1 + 2 * 3,   (1 + 2) * 3 + 4),   ···`

  In the grammar, S is defined as:

  $$S \to T \mid S{+}T$$

Eliminating Ambiguity – Associativity

The unambiguous grammar is:

$$
\begin{aligned}
S &\rightarrow T \mid S\texttt{+}T \\
T &\rightarrow F \mid T\texttt{*}F \\
F &\rightarrow N \mid X \mid \texttt{(}S\texttt{)} \\
N &\rightarrow D \mid DN \\
D &\rightarrow \texttt{0} \mid \cdots \mid \texttt{9} \\
X &\rightarrow \texttt{a} \mid \cdots \mid \texttt{z}
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- This grammar supports the **left-associativity** of + and *. Why?

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- This grammar supports the **left-associativity** of + and $*$. Why?
  - $S \to S\texttt{+}T$ and $T \to T\texttt{*}F$ are **left-recursive**.
- Then, how to support the **right-associativity** of + and $*$?
  - Replace the **left-recursive** rules with **right-recursive** rules!

$$
\begin{array}{rcl}
S & \to & T \mid T\texttt{+}S \\
T & \to & F \mid F\texttt{*}T \\
\cdots & &
\end{array}
$$

Inherent Ambiguity

So far, we have discussed the **ambiguity** for **grammars**.
We will now discuss the **inherent ambiguity** for **languages**.

Definition (Inherent Ambiguity)

A language L is **inherently ambiguous** if all CFGs whose languages are L are ambiguous. (i.e. there is no unambiguous grammar for L)

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For example, the following language is **inherently ambiguous**:

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An example of ambiguous grammar for L is:

$$S \to L \mid R \quad L \to X \mid L\mathtt{c} \quad R \to Y \mid \mathtt{a}R$$
$$X \to \epsilon \mid \mathtt{a}X\mathtt{b} \quad Y \to \epsilon \mid \mathtt{b}Y\mathtt{c}$$

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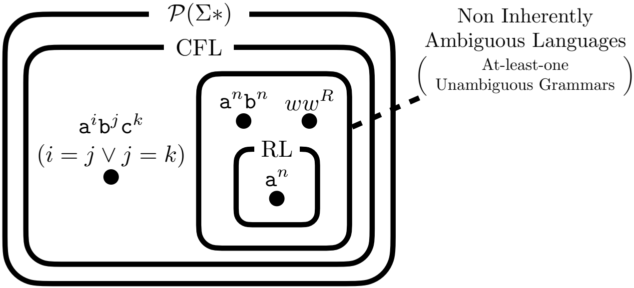
While we can prove that L is inherently ambiguous using the Ogden's lemma[1], we will not discuss it in this course.

[1] https://en.wikipedia.org/wiki/Ogden's_lemma

Summary

1. Parse Trees
   - Definition
   - Yields
   - Relationship between Parse Trees and Derivations
2. Ambiguity
   - Ambiguous Grammars
   - Eliminating Ambiguity
   - Inherent Ambiguity



Midterm Exam

- The midterm exam will be given in class.
- **Date:** 13:30-14:45 (1 hour 15 minutes), April 23 (Wed.).
- **Location:** 301, Aegineung (애기능생활관 301호)
- **Coverage:** Lectures 1 – 13
- **Format:** 7–9 questions with closed book and closed notes
  - Filling blanks in some tables, sentences, or expressions.
  - Construction of automata or grammars for given languages.
  - Proofs of given statements related to languages and automata.
  - Yes/No questions about concepts in the theory of computation.
  - etc.
- Note that there is **no class** on **April 28 (Mon.)**.
- Please refer to the **previous exams** in the course website:

https://plrg.korea.ac.kr/courses/cose215/

Next Lecture

- Pushdown Automata (PDA)

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