Lecture 17 – Deterministic Pushdown Automata (DPDA)

COSE215: Theory of Computation

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PLRG

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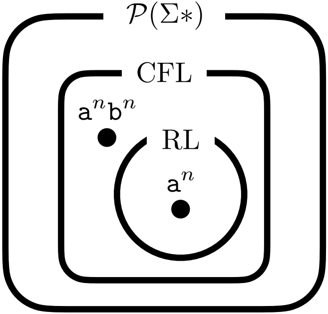
Recall

- A **pushdown automaton (PDA)** is an extension of ϵ-NFA with a **stack**. Thus, PDA is **non-deterministic**.
  - Acceptance by **final states**
  - Acceptance by **empty stacks**
- Then, how about **deterministic PDA (DPDA)**?
- What is the **language class** of DPDA? Still, CFL?



PDA_{FS}
(by final states)
$\|$
PDA_{ES}
(by empty stacks)
$\|$
CFG

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We can check it with the following conditions:

1. $|\delta(q, a, X)| \leq 1$ for all $q \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$.
2. If $\delta(q, \epsilon, X) \neq \varnothing$, then $\delta(q, a, X) = \varnothing$ for all $a \in \Sigma$.

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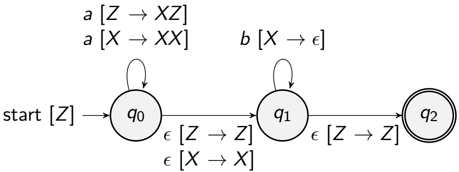

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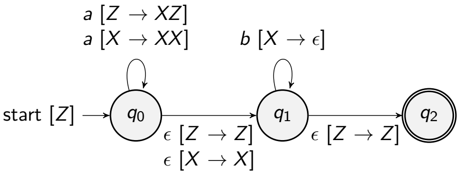

No, because it has multiple transitions for $(q_0, \texttt{ab}, Z)$.

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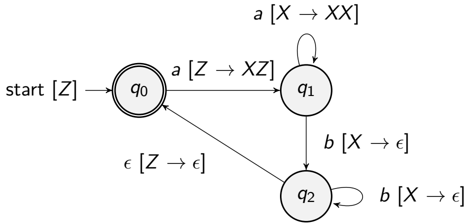

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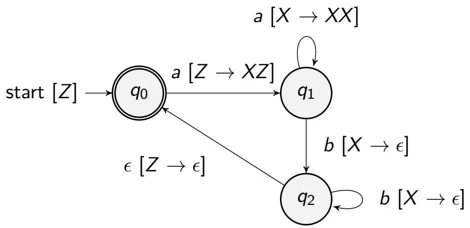

$$
\begin{aligned}
(q_0, \texttt{aabb}, Z) &\vdash (q_1, \texttt{abb}, XZ) \\
&\vdash (q_1, \texttt{bb}, XXZ) \\
&\vdash (q_2, \texttt{b}, XZ) \\
&\vdash (q_2, \epsilon, Z) \\
&\vdash (q_0, \epsilon, \epsilon)
\end{aligned}
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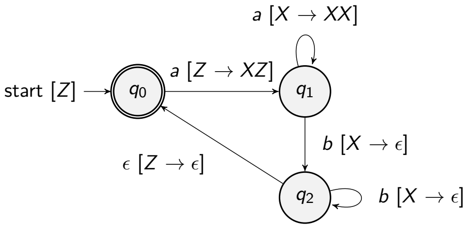

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   - $ww^R = \text{abba} \in L$ where $w = \text{ab}$
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When we read b after ab, we need to consider two possible actions: ① pop Y for b (for abba) or ② push Y for b (for abbbba).

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$$P = (Q, \Sigma, \{Z\}, \delta_P, q_0, Z, F)$$

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$$(q_0, w, Z) \vdash^* (q, \epsilon, Z) \iff \delta^*(q_0, w) = q \qquad \square$$

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❷ DCFL $\setminus$ RL $\neq \varnothing$: We already know that the following language is a DCFL but not an RL:

$$L = \{a^n b^n \mid n \geq 0\} \in \text{DCFL} \setminus \text{RL}$$

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Languages Accepted by Empty Stacks of DPDA (DCFL_{ES})

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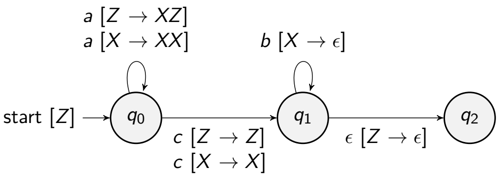

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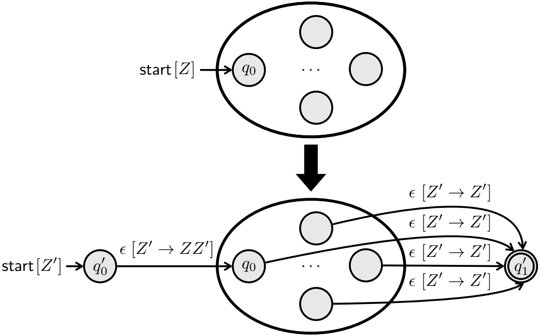

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We can generalize it as **prefix property** of DCFL_{ES}.

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A language L has the **prefix property** if and only if for any word $w \in L$, any proper prefix of w is not in L:

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Thus, L is a **DCFL** but **NOT** a $\mathbf{DCFL_{ES}}$.

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What is the relationship of **inherently ambiguous languages** and DCFLs?

It satisfies the following fact:

$$\text{DCFL} \subsetneq \text{Non Inherently Ambiguous Languages}$$

We prove this fact by the following three steps:

1. $\boxed{\text{DCFL}_{\text{ES}} \subseteq \text{Non Inherently Ambiguous Languages}}$
2. $\boxed{\text{DCFL} \subseteq \text{Non Inherently Ambiguous Languages}}$ (using ①)
3. $\boxed{\text{Non Inherently Ambiguous Languages} \setminus \text{DCFL} \neq \varnothing}$

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1. A DCFL_{ES} has an unambiguous grammar: For a given DCFL_{ES} L and its corresponding DPDA P, we can define a CFG for P as follows:
   - For all $0 \leq j < n$,
   $$S \to A^{Z}_{0,j}$$
   - For all transition $(q_j, X_1 \cdots X_m) \in \delta(q_i, a, X)$ where $q_i \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$ and combinations $0 \leq k_1, \cdots, k_m < n$:
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For any word $w \in L$, w has a unique moves from the initial configuration to the final configuration in P. And, we know that:

$$A^{X}_{i,j} \Rightarrow^* w \qquad \text{if and only if} \qquad (q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$$

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For any word $w \in L$, w has a unique moves from the initial configuration to the final configuration in P. And, we know that:

$$A^{X}_{i,j} \Rightarrow^* w \qquad \text{if and only if} \qquad (q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$$

Thus, the above CFG is **unambiguous**.

Fact 6: DCFL $\subsetneq$ Non Inherently Ambiguous Languages

All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

2. A DCFL has an unambiguous grammar: For a given DCFL L, we can define another DCFL L' with a special symbol \$ as follows:

$$L' = L\$ = \{w\$ \mid w \in L\}$$

PLRG

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For example, $L = \{a^n b^n \mid n \geq 0\}$ is DCFL, then $L' = \{a^n b^n \$ \mid n \geq 0\}$ is a DCFL_{ES} and its **unambiguous grammar** G' is:

$$S \to X\$ \qquad X \to \mathtt{a}X\mathtt{b} \mid \epsilon$$

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Then, the **unambiguous grammar** G for L is:

$$S \to X\$ \qquad X \to \mathtt{a}X\mathtt{b} \mid \epsilon \qquad \$ \to \epsilon$$

PLRG

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All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

3. Non Inherently Ambiguous Languages $\setminus$ DCFL $\neq \varnothing$: The following language is a **non inherently ambiguous language** but **not a DCFL**:

$$L = \{ww^R \mid w \in \{a, b\}^*\}$$

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3. $\boxed{\text{Non Inherently Ambiguous Languages} \setminus \text{DCFL} \neq \varnothing}$: The following language is a **non inherently ambiguous language** but **not a DCFL**:

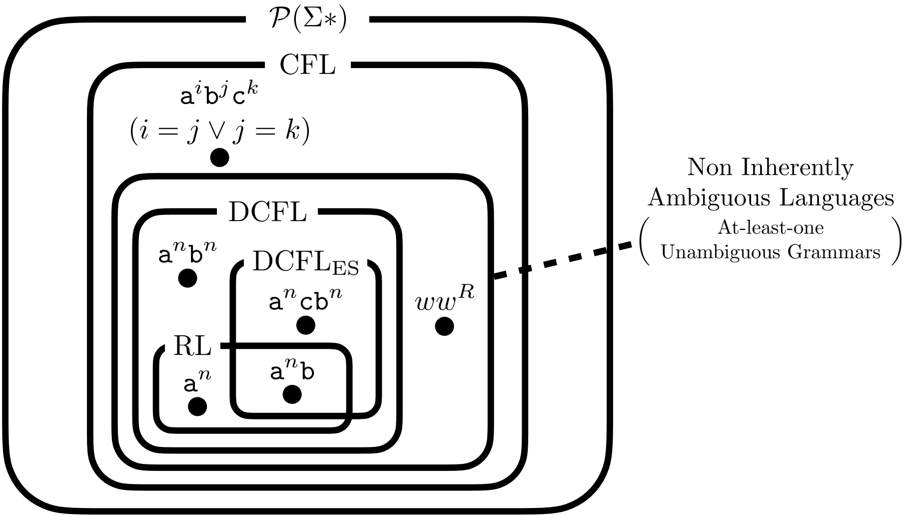
$$L = \{ww^R \mid w \in \{a, b\}^*\}$$

because the following **unambiguous grammar** G represents L:

$$S \rightarrow aSa \mid bSb \mid \epsilon$$

but we already know that L is **not** a **DCFL**.

Summary



$\mathcal{P}(\Sigma*)$
CFL
$a^i b^j c^k$
$(i = j \vee j = k)$
DCFL
$a^n b^n$
DCFL_{ES}
$a^n c b^n$
RL
a^n
$a^n b$
ww^R
Non Inherently Ambiguous Languages
(At-least-one Unambiguous Grammars)

Next Lecture

- Normal Forms of Context-Free Grammars

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