Lecture 23 – Extensions of Turing Machines

COSE215: Theory of Computation

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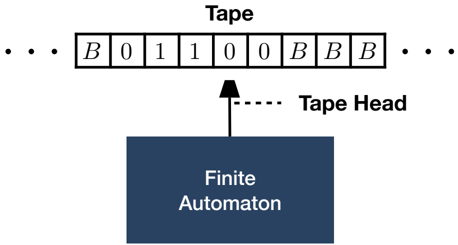
PLRG

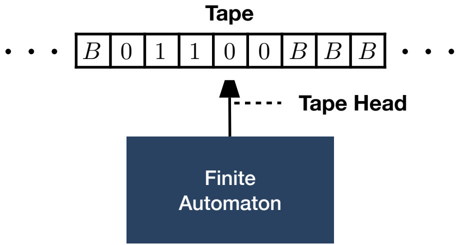
2025 Spring

Recall



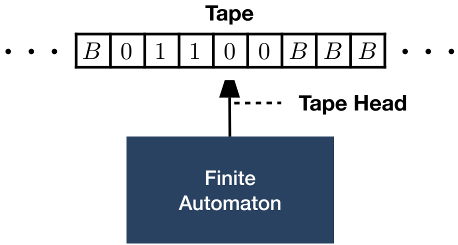
- A **Turing machine (TM)** is a finite automaton with a **tape**.
- A language accepted by a TM is **Recursively Enumerable**.

Recall



- A **Turing machine (TM)** is a finite automaton with a **tape**.
- A language accepted by a TM is **Recursively Enumerable**.
- What happens if we define **other extensions** of TMs?
- Are they **more powerful** than TMs?

Recall

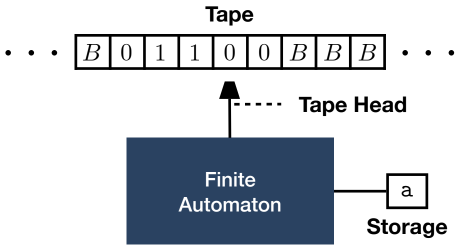


- A **Turing machine (TM)** is a finite automaton with a **tape**.
- A language accepted by a TM is **Recursively Enumerable**.
- What happens if we define **other extensions** of TMs?
- Are they **more powerful** than TMs? **NO!!**

Contents

TMs with Storage

We can define a TM with a **storage**:



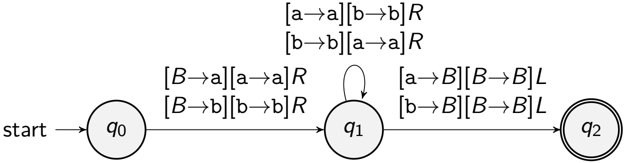
It has additional **storage** affecting the transition function:

$$\delta : Q \times \Gamma \times \Gamma \rightharpoonup Q \times \Gamma \times \Gamma \times \{L, R\}$$

TMs with Storage – Example

$$L(M) = \{\texttt{ab}^n \text{ or } \texttt{ba}^n \mid n \geq 0\}$$

The following **TM with storage** accepts $L(M)$, and see the example for $\texttt{abb} \in L(M)$.[1]



[1] https://plrg.korea.ac.kr/courses/cose215/materials/tm-storage-abn-or-ban.pdf

TMs with Storage are Equivalent to TMs

Theorem

A language accepted by a **TM with storage** *is* **recursively enumerable** *(i.e., accepted by a standard* **TM***).*

TMs with Storage are Equivalent to TMs

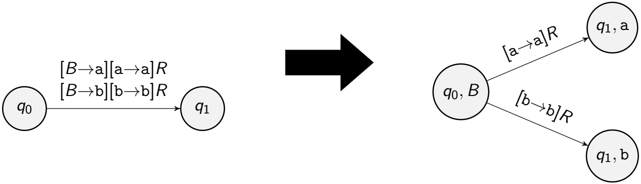
Theorem

A language accepted by a **TM with storage** *is* **recursively enumerable** *(i.e., accepted by a standard* **TM***).*

Proof) We can define an equivalent standard TM by using pairs of states and symbols in the storage as its states:

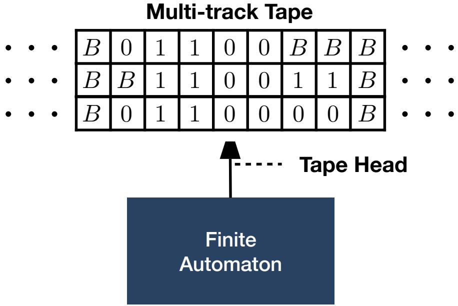
$$\delta'((q, a), b) = \delta(q, a, b)$$

where $Q' = Q \times \Gamma$ and $\delta' : Q' \times \Gamma \rightharpoonup Q' \times \Gamma \times \{L, R\}$.
For example,



Multi-track TMs

We can define a TM with a **multi-track tape**:

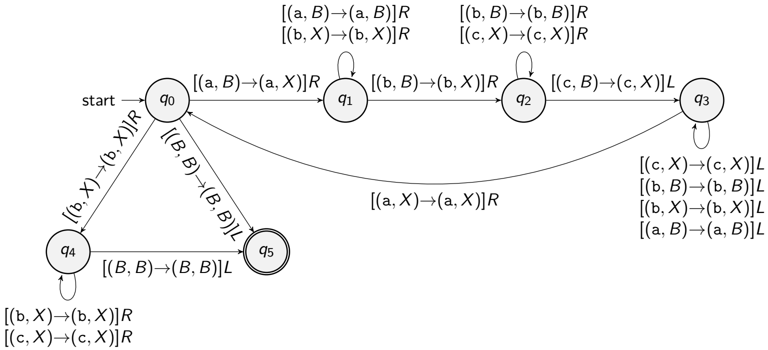


It has a tape with n **tracks** and a **single tape head**:

$$\delta : Q \times \Gamma^{n} \rightharpoonup Q \times \Gamma^{n} \times \{L, R\}$$

$$L(M) = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$$

The following **multi-track TM** accepts $L(M)$, and see the example for $\mathtt{aabbcc} \in L(M)$.[2]



[2] https://plrg.korea.ac.kr/courses/cose215/materials/tm-multi-track-an-bn-cn.pdf

Multi-track TMs are Equivalent to Standard TMs

Theorem

A language accepted by a **multi-track TM** *is* **recursively enumerable** *(i.e., accepted by a standard* **TM***).*

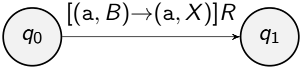
Theorem

A language accepted by a **multi-track TM** *is* **recursively enumerable** *(i.e., accepted by a standard* **TM***).*

Proof) We can define an equivalent standard TM by using n-tuples of symbols as a single symbol:

$$\delta'(q, \alpha) = \delta(q, \alpha)$$

where $\Gamma' = \Gamma^n$ and $\delta' : Q \times \Gamma' \rightharpoonup Q \times \Gamma' \times \{L, R\}$.
For example,

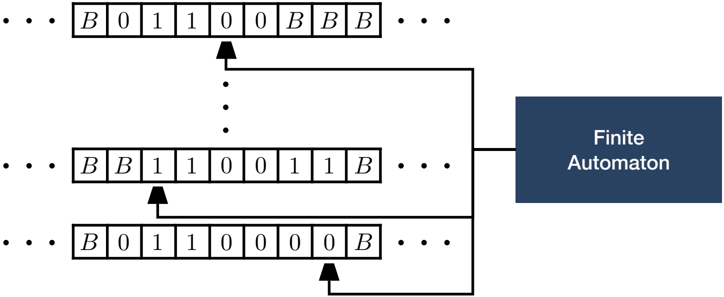


$\cdots$	B	$\mathtt{a}$	$\mathtt{b}$	B	$\cdots$
$\cdots$	B	X	B	B	$\cdots$

$\cdots$	(B, B)	$(\mathtt{a}, X)$	$(\mathtt{b}, B)$	(B, B)	$\cdots$

Multi-tape TMs

We can define a TM with **multiple tapes**:



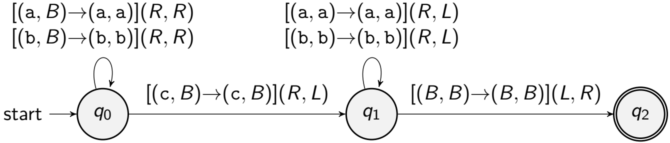
It has n **tapes**, and each tape has its **own head** that can move independently:

$$\delta : Q \times \Gamma^{n} \rightharpoonup Q \times (\Gamma \times \{L, R\})^{n}$$

Multi-tape TMs – Example

$$L(M) = \{w\texttt{c}w^R \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$$

The following **multi-tape TM** accepts $L(M)$, and see the example for $\texttt{abbcbba} \in L(M)$.[3]



[3] https://plrg.korea.ac.kr/courses/cose215/materials/tm-multi-tape-w-c-wr.pdf

Multi-tape TMs are Equivalent to Standard TMs

Theorem

A language accepted by a **multi-tape TM** *is* **recursively enumerable** *(i.e., accepted by a standard* **TM***).*

Multi-tape TMs are Equivalent to Standard TMs

Theorem

A language accepted by a **multi-tape TM** *is* **recursively enumerable** *(i.e., accepted by a standard* **TM***).*

Proof) For a given n-tape TM, we can define an equivalent $2n$-track TM with a storage by using **odd** tracks for the original **tapes** and **even** tracks for the **tape heads**:

$\cdots$	B	a	b	b	B	$\cdots$
		$\uparrow$				
$\cdots$	B	b	c	a	B	$\cdots$
				$\uparrow$		

$\Rightarrow$

$\cdots$	B	a	b	b	B	$\cdots$
$\cdots$	B	X	B	B	B	$\cdots$
$\cdots$	B	b	c	a	B	$\cdots$
$\cdots$	B	B	B	X	B	$\cdots$

We can simulate one-step in the n-tape TM by 1) scanning the tape to store the symbols under the n heads into the storage, and then 2) scanning the tape again to update the symbols and move the heads.

Multi-tape TMs are Equivalent to Standard TMs

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A language accepted by a **multi-tape TM** *is* **recursively enumerable** *(i.e., accepted by a standard* **TM***).*

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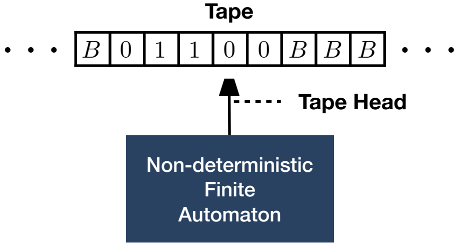
$\cdots$	B	X	B	B	B	$\cdots$
$\cdots$	B	b	c	a	B	$\cdots$
$\cdots$	B	B	B	X	B	$\cdots$

We can simulate one-step in the n-tape TM by 1) scanning the tape to store the symbols under the n heads into the storage, and then 2) scanning the tape again to update the symbols and move the heads.

However, it is **inefficient** because we need to scan all the symbols on the tape to simulate a single step in the n-tape TM.

Non-deterministic TMs (NTMs)

We can define a TM with **non-deterministic transitions**:



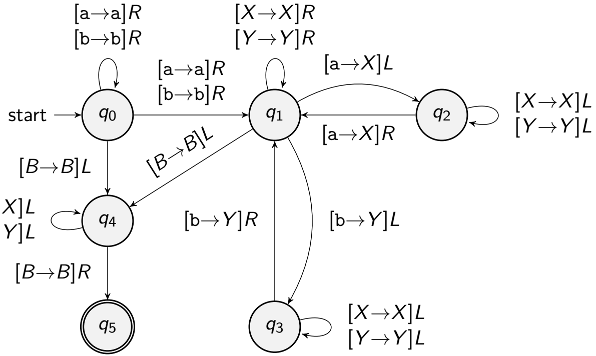
It has a **non-deterministic transition function**:

$$\delta : Q \times \Gamma \to \mathcal{P}(\, Q \times \Gamma \times \{L, R\} \,)$$

Non-deterministic TMs – Example

$$L(M) = \{ww^R \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$$

The following **nondeterministic TM** accepts $L(M)$, and see the example for $\mathtt{abba} \in L(M)$.[4]



[4] https://plrg.korea.ac.kr/courses/cose215/materials/ntm-w-wr.pdf

Non-deterministic TMs are Equivalent to Standard TMs

Theorem

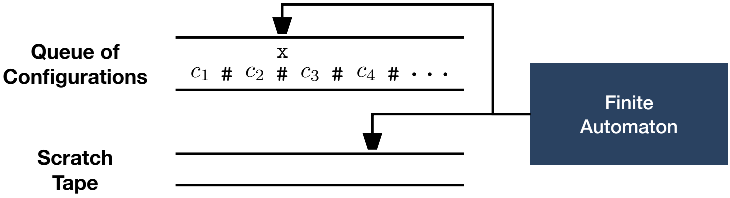
A language accepted by a **non-deterministic TM** *is* **recursively enumerable** *(i.e., accepted by a standard* **TM***).*

Non-deterministic TMs are Equivalent to Standard TMs

Theorem

A language accepted by a **non-deterministic TM** *is* **recursively enumerable** *(i.e., accepted by a standard* **TM***).*

Proof) For a given non-deterministic TM, we can define an equivalent 2-tape TM: 1) a 2-track tape to maintain a **queue of configurations** and 2) a normal track to **simulate** the tape of the original TM.



Non-deterministic TMs are Equivalent to Standard TMs

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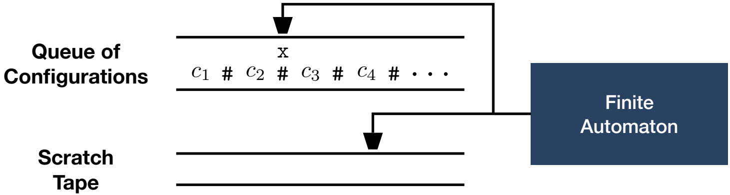
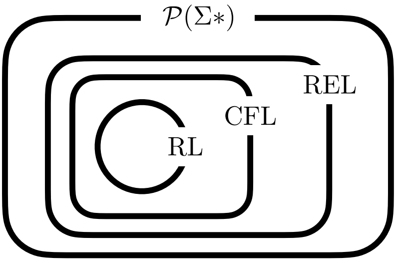


Similarly, it is **inefficient** because we need to capture all the configurations of the non-deterministic TM to simulate a single step.

More Extensions of TMs

- There are more extensions of TMs:
  - TMs with **Stay Option** – L: Left, R: Right, and S: **Stay**
  - **Queue Automata** – Automata with **Queue**
  - **Random Access Machines** – TMs with **Random Access Memory**
  - **Quantum TMs** – TMs with **Quantum States**
  - . . .
- They are all **equivalent** to TMs.
- A standard **TM** is the **most powerful model of computation**.



TM = ETM

(Turing Machine) (All Extensions of TMs)

TM = REL

(Recursively Enumerable Language)

Summary

1. Extensions of Turing Machines
    - TMs with Storage
    - Multi-track TMs
    - Multi-tape TMs
    - Non-deterministic TMs (NTMs)
    - More Extensions of TMs

Homework #6

- Please see this document on GitHub:

https://github.com/ku-plrg-classroom/docs/tree/main/cose215/tm-examples

- The due date is 23:59 on Jun. 16 (Mon.).
- Please only submit `Implementation.scala` file to **LMS**.

Next Lecture

- The Origin of Computer Science

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