Lecture 24 – The Origin of Computer Science

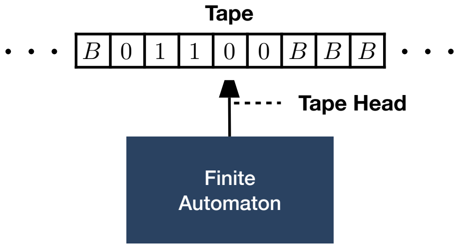
COSE215: Theory of Computation

Jihyeok Park

PLRG

2025 Spring

Recall



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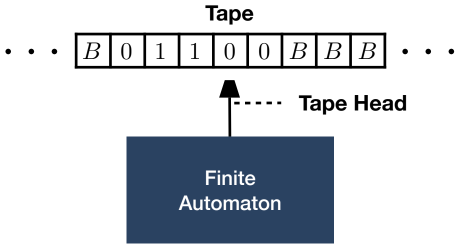

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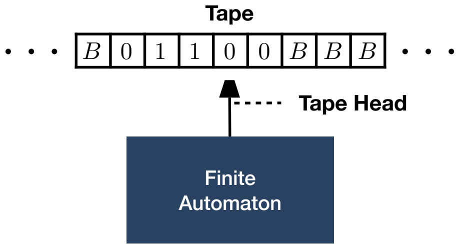

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- Why is TM the **origin of Computer Science**?

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Gödel's Incompleteness Theorem

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(1862 – 1943)

I argue that any statement is **True** or **False**, not both (*Consistent*), and we can **PROVE** that any **True** statement (*Complete*)!

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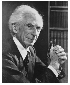

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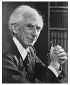

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(e.g., **ZFC** - **Z**ermelo–**F**raenkel set theory with Axiom of **C**hoice)

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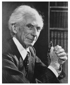

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1st Gödel's Incompleteness Theorem (1931)

Unfortunately, I proved that there always exists a statement that is **True** but **Unprovable** under **any set of axioms**.

Kurt Gödel
(1906 – 1978)

Example: Continuum Hypothesis

- **Cardinality**: The number of elements in a set.

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  - The set of **non-negative even numbers** is **countably infinite**.

$$\mathbb{N} \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \{n \in \mathbb{N} \mid n \geq 0 \land n \equiv 0 \pmod 2\} \text{ where } f(n) = 2n \text{ and } f^{-1}(n) = \frac{n}{2}$$

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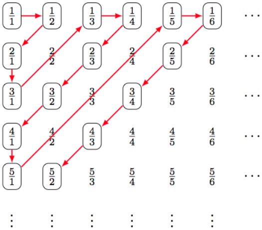
  - The set of **rational numbers** is **countably infinite**.



Example: Continuum Hypothesis

- A set of **real numbers** between 0 and 1 is **uncountably infinite** and its cardinality ($\aleph_1 = 2^{\aleph_0}$) is strictly larger than the set of natural numbers ($\aleph_1 > \aleph_0$) because of **Cantor's diagonal argument**:

n	$f(n)$											
1	0	.	**3**	1	4	1	5	9	2	6	5	3	…
2	0	.	3	**7**	3	7	3	7	3	7	3	7	…
3	0	.	1	4	**2**	8	5	7	1	4	2	8	…
4	0	.	7	0	7	**1**	0	6	7	8	1	1	…
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- **Continuum Hypothesis**: There is no set whose cardinality is strictly between $\aleph_0$ and $\aleph_1$:

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- Kurt Gödel and Paul Cohen showed we **CANNOT** either prove or disprove the **Continuum Hypothesis** using the standard axioms of set theory, **ZFC** (**Z**ermelo-**F**raenkel set theory with the **A**xiom of **C**hoice).

Gödel Numbering

- **Gödel Numbering**: Assign a unique number to each symbol and string in a formal language.

Symbol	$\sim$	$\vee$	$\supset$	$\exists$	$=$	0	s	(	)	,	$+$
Number	1	2	3	4	5	6	7	8	9	10	11
Symbol	$\times$	x	y	z	p	q	r	P	Q	R	
Number	12	13	14	15	16	17	18	19	20	21	

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- We will use **prime numbers** to encode strings:

$$\Gamma(x_1 \cdots x_n) = \prod_{i=1}^{n} p_i^{x_i}$$

where p_i is the i-th prime number.

- For example, $\Gamma(\texttt{0=0}) = 2^6 \times 3^5 \times 5^6 = 243{,}000{,}000$.

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- For example, $\Gamma(0{=}0) = 2^6 \times 3^5 \times 5^6 = 243,000,000$.
- Gödel used this idea to encode **formulas** and **proofs** in **first-order arithmetic**, and then proved his famous **Incompleteness Theorem**.[1]

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$$\forall x \forall y.\ x\ \mathbf{Dem}\ y \text{ iff } \Gamma^{-1}(x) \text{ is a proof of } \Gamma^{-1}(y)$$

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This is **true** but **not provable** in any consistent mathematical system.[2]

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- **Turing Machine** is the origin of **computers**.
- **Lambda Calculus** is the origin of **programming languages**.

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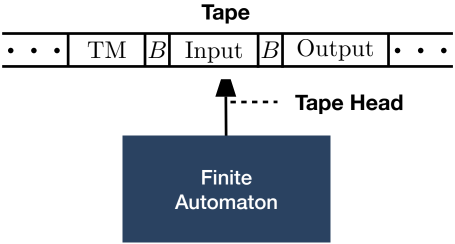

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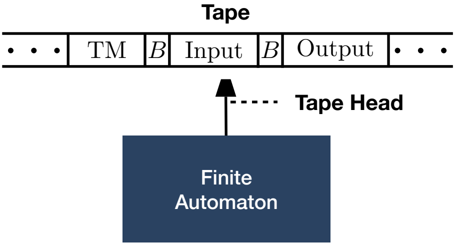

- **UTM** was **the most important invention in computer science** because it was the first time we can write a **program (software)** instead of building a new **machine (hardware)** to solve a new problem.

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$$\forall \text{ TM } M.\ \forall w \in a^*.\ H(M, w) = \begin{cases} \texttt{halt} & \text{if } A(\text{``}M \text{ halts on } w\text{''}) \\ \texttt{loop} & \text{otherwise} \end{cases}$$

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- Then, F is not in the enumeration (i.e., $F \neq M_i$ for all i). It contradicts the **enumerability of TMs**. So, A **does not exist.**

Lambda Calculus

- **Alonzo Church**'s definition of computation is the **Lambda Calculus (LC)**:

$$
\begin{array}{rcll}
\Lambda \ni E & ::= & x & \text{(Variable)} \\
& \mid & \lambda x.\ E & \text{(Abstraction)} \\
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- A **computable function** is a **lambda term**.
- If there is no more possible β-reduction, the term is in **normal form**.

Lambda Calculus – Church Encoding

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- Surprisingly, we can **encode** them – **Church Encoding**:

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Boolean Values and Operations

$$\begin{aligned}\text{true} &= \lambda x.\ \lambda y.\ x\\ \text{false} &= \lambda x.\ \lambda y.\ y\\ \text{and} &= \lambda b_1.\ \lambda b_2.\ b_1\ b_2\ \text{false}\\ \text{or} &= \lambda b_1.\ \lambda b_2.\ b_1\ \text{true}\ b_2\end{aligned}$$

Natural Numbers and Operations

$$\begin{aligned}0 &= \lambda f.\ \lambda x.\ x\\ 1 &= \lambda f.\ \lambda x.\ f\ x\\ 2 &= \lambda f.\ \lambda x.\ f\ (f\ x)\\ 3 &= \lambda f.\ \lambda x.\ f\ (f\ (f\ x))\\ \text{plus} &= \lambda n_1.\ \lambda n_2.\ \lambda f.\ \lambda x.\ n_1\ f\ (n_2\ f\ x)\\ \text{times} &= \lambda n_1.\ \lambda n_2.\ \lambda f.\ \lambda x.\ n_1\ (n_2\ f)\ x\\ \text{exp} &= \lambda n_1.\ \lambda n_2.\ n_2\ n_1\end{aligned}$$

Control Flows

$$\begin{aligned}\text{if} &= \lambda b.\ \lambda e_1.\ \lambda e_2.\ b\ e_1\ e_2\\ \text{Y} &= \lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))\end{aligned}$$

Pairs

$$\begin{aligned}\text{pair} &= \lambda x.\ \lambda y.\ \lambda f.\ f\ x\ y\\ \text{fst} &= \lambda p.\ p\ (\lambda x.\ \lambda y.\ x)\\ \text{snd} &= \lambda p.\ p\ (\lambda x.\ \lambda y.\ y)\end{aligned}$$

Lists

$$\begin{aligned}\text{nil} &= \lambda c.\ \lambda n.\ n\\ \text{cons} &= \lambda h.\ \lambda t.\ \lambda c.\ \lambda n.\ c\ h\ (t\ c\ n)\\ \text{head} &= \lambda l.\ l\ (\lambda h.\ \lambda t.\ h)\\ \text{isnil} &= \lambda l.\ l\ (\lambda h.\ \lambda t.\ \text{false})\ \text{true}\end{aligned}$$

Lambda Calculus – Church Encoding

PLRG

$$
\begin{aligned}
0 &= \lambda f.\ \lambda x.\ x \\
1 &= \lambda f.\ \lambda x.\ f\ x \\
2 &= \lambda f.\ \lambda x.\ f\ (f\ x) \\
3 &= \lambda f.\ \lambda x.\ f\ (f\ (f\ x))
\end{aligned}
$$

$$
\begin{aligned}
\text{plus} &= \lambda n_1.\ \lambda n_2.\ \lambda f.\ \lambda x.\ n_1\ f\ (n_2\ f\ x) \\
\text{times} &= \lambda n_1.\ \lambda n_2.\ \lambda f.\ \lambda x.\ n_1\ (n_2\ f)\ x \\
\text{exp} &= \lambda n_1.\ \lambda n_2.\ n_2\ n_1
\end{aligned}
$$

For example, we can compute $1 + 1$ as follows:

$$
\begin{aligned}
\text{plus}\ 1\ 1 \quad &= \quad (\lambda n_1.\ \lambda n_2.\ \lambda f.\ \lambda x.\ n_1\ f\ (n_2\ f\ x))\ 1\ 1 \\
&\to \quad \lambda f.\ \lambda x.\ 1\ f\ (1\ f\ x) \\
&= \quad \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda f.\ \lambda x.\ f\ x)\ f\ x) \\
&\to \quad \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (f\ x) \\
&\to \quad \lambda f.\ \lambda x.\ f\ (f\ x) \\
&= \quad 2
\end{aligned}
$$

The **normal form** (computational result) of (plus 1 1) is 2.

Disproof using Lambda Calculus

- Church proved that there is **no computable function** that can decide whether two **lambda terms** are **equivalent** or not:

$$\exists\ \text{eq?} \in \Lambda.\ \forall\ E_1, E_2 \in \Lambda.\ (\text{eq?}\ E_1\ E_2) \rightarrow \begin{cases} \text{true} & \text{if } E_1 \equiv E_2 \\ \text{false} & \text{otherwise} \end{cases}$$

  where $E_1 \equiv E_2$ means E_1 and E_2 are equivalent, i.e., they have the same **normal form** (computational result).

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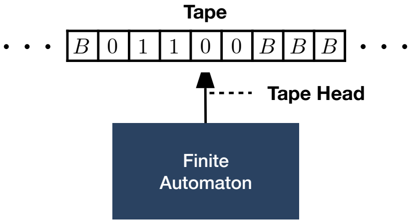
  where $E_1 \equiv E_2$ means E_1 and E_2 are equivalent, i.e., they have the same **normal form** (computational result).
- For example, (plus 1 1) and (plus 0 2) are equivalent in LC because they have the same normal form 2.
- It means that there is no computable function that can **decide** whether a **lambda term** has a given **normal form** or not.
- We skip the proof here.

Contents

Church-Turing Thesis



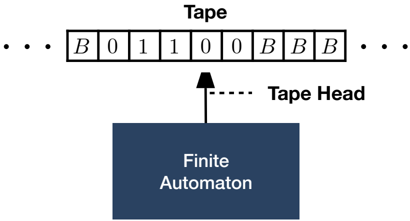
(Turning Machine)

$$\equiv \quad \Lambda \ni E \;::=\; x \quad \text{(Variable)} \;\mid\; \lambda x.\, E \quad \text{(Abstraction)} \;\mid\; E\ E \quad \text{(Application)}$$

(Lambda Calculus)

- **LC** has the same computational power as **TMs**. (**Turing Complete**)

Church-Turing Thesis



(Turning Machine)

$\equiv$

$$
\begin{array}{rcll}
\Lambda \ni E & ::= & x & \text{(Variable)} \\
& \mid & \lambda x.\ E & \text{(Abstraction)} \\
& \mid & E\ E & \text{(Application)}
\end{array}
$$

(Lambda Calculus)

- **LC** has the same computational power as **TMs**. (**Turing Complete**)
- **Church-Turing Thesis**:

  Any **real-world computation** *can be translated into an equivalent computation involving a* **Turing machine** *or can be done using* **lambda calculus**.

Summary

1. Gödel's Incompleteness Theorem
   - Example: Continuum Hypothesis
   - Gödel Numbering

2. Entscheidungsproblem – Decision Problem
   - Disproof using Turing Machine
   - Disproof using Lambda Calculus

3. Church-Turing Thesis

Next Lecture

- Undecidability

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