Lecture 9 – The Pumping Lemma for Regular Languages

COSE215: Theory of Computation

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PLRG

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Recall

The class of regular languages is closed under the following operations:

- Union
- Concatenation
- Kleene Star
- Complement
- Intersection
- Difference
- Reversal
- Homomorphism
- Inverse Homomorphism

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For example, consider $L_i = \{\mathtt{a}^i\mathtt{b}^i\}$ for $i \geq 0$. Then, L_i and its complement L_i^c are regular languages. But, their infinite union and intersection are not regular languages:

$$\bigcup_{i \geq 0} L_i = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\} = L \qquad \bigcap_{i \geq 0} L_i^c = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}^c = L^c$$

because $L = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$ is not regular.

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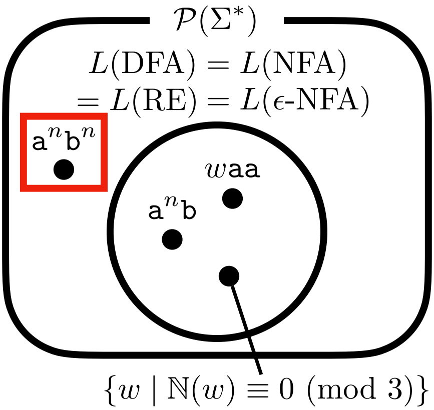
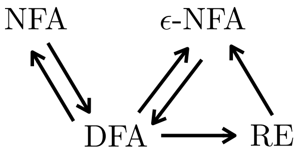
- Not all languages are regular: e.g., $L = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$.





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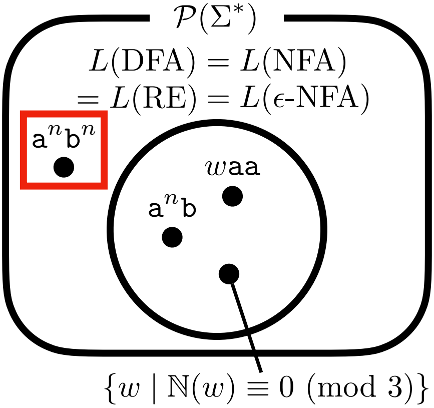

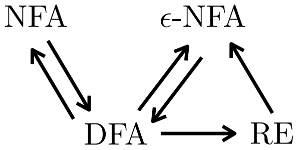

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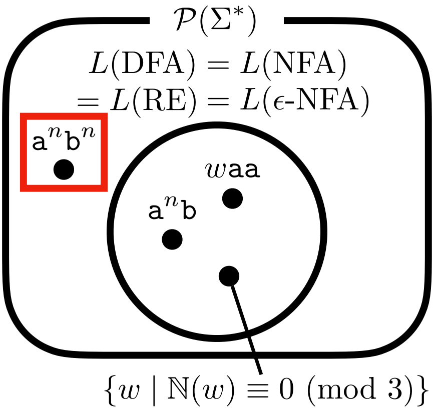

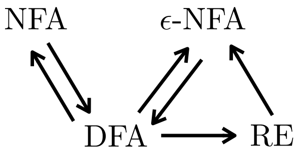

- How to prove that a language is **NOT** regular? **Pumping Lemma!**

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   The **Pumping Lemma** formally captures this intuition.

Pumping Lemma for Regular Languages

Lemma (Pumping Lemma for Regular Languages)

For a given regular language L, **there exists** *a positive integer n such that* **for all** $w \in L$, *if* $|w| \geq n$, **there exists** $w = xyz$ *such that*

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$$q_0 = p_0 \xrightarrow{\mathtt{a}_1} p_1 \xrightarrow{\mathtt{a}_2} \cdots \xrightarrow{\mathtt{a}_n} p_n \xrightarrow{\mathtt{a}_{n+1}} \cdots \xrightarrow{\mathtt{a}_m} p_m \in F$$

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$$1 - |y| = j - i > 0 \qquad\qquad |xy| = j \leq n - 2$$

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A & = & L \text{ is regular} \\
& & \Downarrow \\
B & = & \exists n > 0.\ \forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③
\end{array}
$$

$$
\begin{array}{rcll}
A & \Longrightarrow & B & \text{(O)} \\
B & \Longrightarrow & A & \text{(X)} \\
\neg B & \Longrightarrow & \neg A & \text{(O)}
\end{array}
$$

$$
\begin{array}{rcl}
\neg B & = & \neg(\exists n > 0.\ \forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③) \\
& = & \forall n > 0.\ \neg(\forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③) \\
& = & \forall n > 0.\ \exists w \in L.\ \neg(|w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③) \\
& = & \forall n > 0.\ \exists w \in L.\ |w| \geq n \land \neg(\exists w = xyz.\ ① \land ② \land ③) \\
& = & \forall n > 0.\ \exists w \in L.\ |w| \geq n \land \forall w = xyz.\ \neg(① \land ② \land ③)
\end{array}
$$

Lemma (Pumping Lemma for Regular Languages)

$$
\begin{array}{rcc}
A & = & L \text{ is regular} \\
 & & \Downarrow \\
B & = & \exists n > 0.\ \forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \wedge ② \wedge ③
\end{array}
$$

$$
\begin{array}{rcll}
A & \Longrightarrow & B & (\mathrm{O}) \\
B & \Longrightarrow & A & (\mathrm{X}) \\
\neg B & \Longrightarrow & \neg A & (\mathrm{O})
\end{array}
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$$
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 & = & \forall n > 0.\ \neg(\forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \wedge ② \wedge ③) \\
 & = & \forall n > 0.\ \exists w \in L.\ \neg(|w| \geq n \Rightarrow \exists w = xyz.\ ① \wedge ② \wedge ③) \\
 & = & \forall n > 0.\ \exists w \in L.\ |w| \geq n \wedge \neg(\exists w = xyz.\ ① \wedge ② \wedge ③) \\
 & = & \forall n > 0.\ \exists w \in L.\ |w| \geq n \wedge \forall w = xyz.\ \neg(① \wedge ② \wedge ③) \\
 & = & \forall n > 0.\ \exists w \in L.\ |w| \geq n \wedge \forall w = xyz.\ \neg(① \wedge ②) \vee \neg ③
\end{array}
$$

Application: Proving Languages are Not Regular

Lemma (Pumping Lemma for Regular Languages)

$$A \;=\; L \textit{ is regular}$$
$$\Downarrow$$
$$B \;=\; \exists n > 0.\ \forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③$$

$$\begin{array}{rcll} A & \Longrightarrow & B & \text{(O)} \\ B & \Longrightarrow & A & \text{(X)} \\ \neg B & \Longrightarrow & \neg A & \text{(O)} \end{array}$$

$$\begin{aligned}
\neg B &= \neg(\exists n > 0.\ \forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③) \\
&= \forall n > 0.\ \neg(\forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③) \\
&= \forall n > 0.\ \exists w \in L.\ \neg(|w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③) \\
&= \forall n > 0.\ \exists w \in L.\ |w| \geq n \land \neg(\exists w = xyz.\ ① \land ② \land ③) \\
&= \forall n > 0.\ \exists w \in L.\ |w| \geq n \land \forall w = xyz.\ \neg(① \land ② \land ③) \\
&= \forall n > 0.\ \exists w \in L.\ |w| \geq n \land \forall w = xyz.\ \neg(① \land ②) \lor \neg ③ \\
&= \forall n > 0.\ \exists w \in L.\ |w| \geq n \land \forall w = xyz.\ (① \land ②) \Rightarrow \neg ③
\end{aligned}$$

Application: Proving Languages are Not Regular

To prove a language L is **NOT** regular, we need to show that

$$\forall n > 0.\ \exists w \in L.\ |w| \geq n \land \forall w = xyz.\ (①\land ②) \Rightarrow \neg ③$$

1. $|y| > 0$
2. $|xy| \leq n$
3. $\forall i \geq 0.\ xy^iz \in L$

Note that $\neg ③ = \exists i \geq 0.\ xy^iz \notin L$.

Application: Proving Languages are Not Regular

To prove a language L is **NOT** regular, we need to show that

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1. $|y| > 0$
2. $|xy| \leq n$
3. $\forall i \geq 0.\ xy^iz \in L$

Note that $\neg ③ = \exists i \geq 0.\ xy^iz \notin L$.

We can prove this by following the steps below:

1. Assume any positive integer n is given.
2. Pick a word $w \in L$.
3. Show that $|w| \geq n$.
4. Assume any split $w = xyz$ is given, and ① $|y| > 0 \land$ ② $|xy| \leq n$.
5. $\neg$③ Pick $i \geq 0$, and show that $xy^iz \notin L$ using ① and ②.

Contents

Example 1

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1. Assume any positive integer n is given.

Example 1

Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$$

1. Assume any positive integer n is given.
2. Let $w = \mathtt{a}^n\mathtt{b}^n \in L$.
3. $|w| = n + n = 2n \geq n$.
4. Assume any split $w = xyz$ is given, and 1 $|y| > 0 \land$ 2 $|xy| \leq n$.

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3. $|w| = n + n = 2n \geq n$.
4. Assume any split $w = xyz$ is given, and ① $|y| > 0 \land$ ② $|xy| \leq n$.
5. Let $i = \boxed{0}$. We need to show that $\neg$③ $xy^0z \notin L$:
   - Since ② $|xy| \leq n$,

     $$x = \mathtt{a}^p \qquad y = \mathtt{a}^q \qquad z = \mathtt{a}^{n-p-q}\mathtt{b}^n$$

     for some $0 \leq p, q \leq n$ such that $p + q \leq n$.
   - Since ① $|y| > 0$, we know $q > 0$.
   - Finally, $xy^0z = xz = \mathtt{a}^p\mathtt{a}^{n-p-q}\mathtt{b}^n = \mathtt{a}^{n-q}\mathtt{b}^n \notin L$ ($\because q > 0$). $\square$

Example 2

Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{ww^R \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$$

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Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{ww^R \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$$

1. Assume any positive integer n is given.
2. Let $w = \mathtt{a}^n\mathtt{b}^n\mathtt{b}^n\mathtt{a}^n \in L$.
3. $|w| = n + n + n + n = 4n \geq n$.
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3. $|w| = n + n + n + n = 4n \geq n$.
4. Assume any split $w = xyz$ is given, and 1 $|y| > 0 \land$ 2 $|xy| \leq n$.
5. Let $i = 0$. We need to show that $\neg$3 $xy^0z \notin L$:
   - Since 2 $|xy| \leq n$,

     $$x = \mathtt{a}^p \qquad y = \mathtt{a}^q \qquad z = \mathtt{a}^{n-p-q}\mathtt{b}^n\mathtt{b}^n\mathtt{a}^n$$

     for some $0 \leq p, q \leq n$ such that $p + q \leq n$.
   - Since 1 $|y| > 0$, we know $q > 0$.
   - Finally, $xy^0z = xz = \mathtt{a}^p\mathtt{a}^{n-p-q}\mathtt{b}^n\mathtt{b}^n\mathtt{a}^n = \mathtt{a}^{n-q}\mathtt{b}^n\mathtt{b}^n\mathtt{a}^n \notin L$ $(\because q > 0)$.

$\square$

Example 3

Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{\mathtt{a}^l \mathtt{b}^m \mathtt{c}^n \mid l + m \leq n\}$$

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1. Assume any positive integer n is given.

Example 3

PLRG

Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{\mathtt{a}^l \mathtt{b}^m \mathtt{c}^n \mid l + m \leq n\}$$

1. Assume any positive integer n is given.
2. Let $w = \mathtt{a}^n \mathtt{b}^n \mathtt{c}^{2n} \in L$.
3. $|w| = n + n + 2n = 4n \geq n$.
4. Assume any split $w = xyz$ is given, and ① $|y| > 0 \land$ ② $|xy| \leq n$.

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Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{\mathtt{a}^l \mathtt{b}^m \mathtt{c}^n \mid l + m \leq n\}$$

1. Assume any positive integer n is given.
2. Let $w = \mathtt{a}^n \mathtt{b}^n \mathtt{c}^{2n} \in L$.
3. $|w| = n + n + 2n = 4n \geq n$.
4. Assume any split $w = xyz$ is given, and ① $|y| > 0 \land$ ② $|xy| \leq n$.
5. Let $i =$ 2 . We need to show that $\neg$③ $xy^2z \notin L$:
   - Since ② $|xy| \leq n$,

     $$x = \mathtt{a}^p \qquad y = \mathtt{a}^q \qquad z = \mathtt{a}^{n-p-q} \mathtt{b}^n \mathtt{c}^{2n}$$

     for some $0 \leq p, q \leq n$ such that $p + q \leq n$.
   - Since ① $|y| > 0$, we know $q > 0$.
   - Finally, $xy^2z = xyyz = \mathtt{a}^{n+q} \mathtt{b}^n \mathtt{c}^{2n} \notin L$
     ($\because q > 0$. Thus, $(n + q) + n = 2n + q > 2n$). □

Example 4

Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{\mathtt{a}^{n^2} \mid n \geq 0\}$$

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$$L = \{\mathtt{a}^{n^2} \mid n \geq 0\}$$

1. Assume any positive integer n is given.
2. Let $w = \mathtt{a}^{n^2} \in L$.
3. $|w| = n^2 \geq n$.
4. Assume any split $w = xyz$ is given, and ① $|y| > 0 \land$ ② $|xy| \leq n$.

Example 4

Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{\mathtt{a}^{n^2} \mid n \geq 0\}$$

1. Assume any positive integer n is given.
2. Let $w = \mathtt{a}^{n^2} \in L$.
3. $|w| = n^2 \geq n$.
4. Assume any split $w = xyz$ is given, and 1 $|y| > 0 \wedge$ 2 $|xy| \leq n$.
5. Let $i = 2$. We need to show that $\neg$3 $xy^2z \notin L$:
   - Since 1 $|y| > 0$ and 2 $|xy| \leq n$,

     $$y = \mathtt{a}^k$$

     where $0 < k \leq n$. Then,

     $$n^2 < n^2 + k \;(\because 0 < k) \qquad n^2 + k < (n+1)^2 \;(\because k \leq n)$$

   - Finally, $xy^2z = xyyz = \mathtt{a}^{n^2+k} \notin L$ □

Example 5

Let's prove that L is **NOT** regular:

$$L = \{\mathtt{a}^n\mathtt{b}^k\mathtt{c}^{n+k} \mid n, k \geq 0\}$$

Example 5

Let's prove that L is **NOT** regular:

$$L = \{\mathtt{a}^n \mathtt{b}^k \mathtt{c}^{n+k} \mid n, k \geq 0\}$$

- It is much easier to use **closure properties** under **homomorphisms**.
- Consider a homomorphism $h : \{\mathtt{a}, \mathtt{b}, \mathtt{c}\} \to \{\mathtt{a}, \mathtt{b}\}^*$:

$$h(\mathtt{a}) = \mathtt{a} \qquad h(\mathtt{b}) = \mathtt{a} \qquad h(\mathtt{c}) = \mathtt{b}$$

- Then,

$$h(L) = \{\mathtt{a}^{n+k} \mathtt{b}^{n+k} \mid n, k \geq 0\} = \{\mathtt{a}^n \mathtt{b}^n \mid n \geq 0\}$$

- If L is regular, then $h(L)$ must be regular as well.
- However, we know $h(L)$ is **NOT** regular.
- Therefore, L is **NOT** regular. $\square$

Summary

1. Pumping Lemma for Regular Languages
   - Intuition
   - Pumping Lemma
   - Proof of Pumping Lemma
   - Application: Proving Languages are Not Regular

2. Examples
   - Example 1: $L = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$
   - Example 2: $L = \{ww^R \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$
   - Example 3: $L = \{\mathtt{a}^l\mathtt{b}^m\mathtt{c}^n \mid l + m \leq n\}$
   - Example 4: $L = \{\mathtt{a}^{n^2} \mid n \geq 0\}$
   - Example 5: $L = \{\mathtt{a}^n\mathtt{b}^k\mathtt{c}^{n+k} \mid n, k \geq 0\}$

Homework #3

- Please see this document on GitHub:

https://github.com/ku-plrg-classroom/docs/tree/main/cose215/equiv-re-fa

- The due date is 23:59 on Apr. 21 (Mon.).
- Please implement the following functions in **`Implementation.scala`**.
  - `reToENFA` for the conversion from REs to ϵ-NFAs.
  - `dfaToRE` for the conversion from DFAs to REs.
  - `enfaToDFA` for the conversion from ϵ-NFAs to DFAs.
- Please only submit **`Implementation.scala`** file to **LMS**.

Next Lecture

PLRG

- Equivalence and Minimization of Finite Automata

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