Final Exam

COSE215: Theory of Computation

2026 Spring

Instructor: Jihyeok Park

June 17, 2026. 13:30-14:45

- **If you are not good at English, please write your answers in Korean.**
  (영어가 익숙하지 않은 경우, 답안을 한글로 작성해 주세요.)
- **Write answers in good handwriting.**
  If we cannot recognize your answers, you will not get any points.
  (글씨를 알아보기 힘들면 점수를 드릴 수 없습니다. 답안을 읽기 좋게 작성해주세요.)
- **Write your answers in the boxes provided.**
  (답안을 제공된 박스 안에 작성해 주세요.)

Student ID	
Student Name	

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	15	10	10	15	15	10	10	8	7	100
Score:										

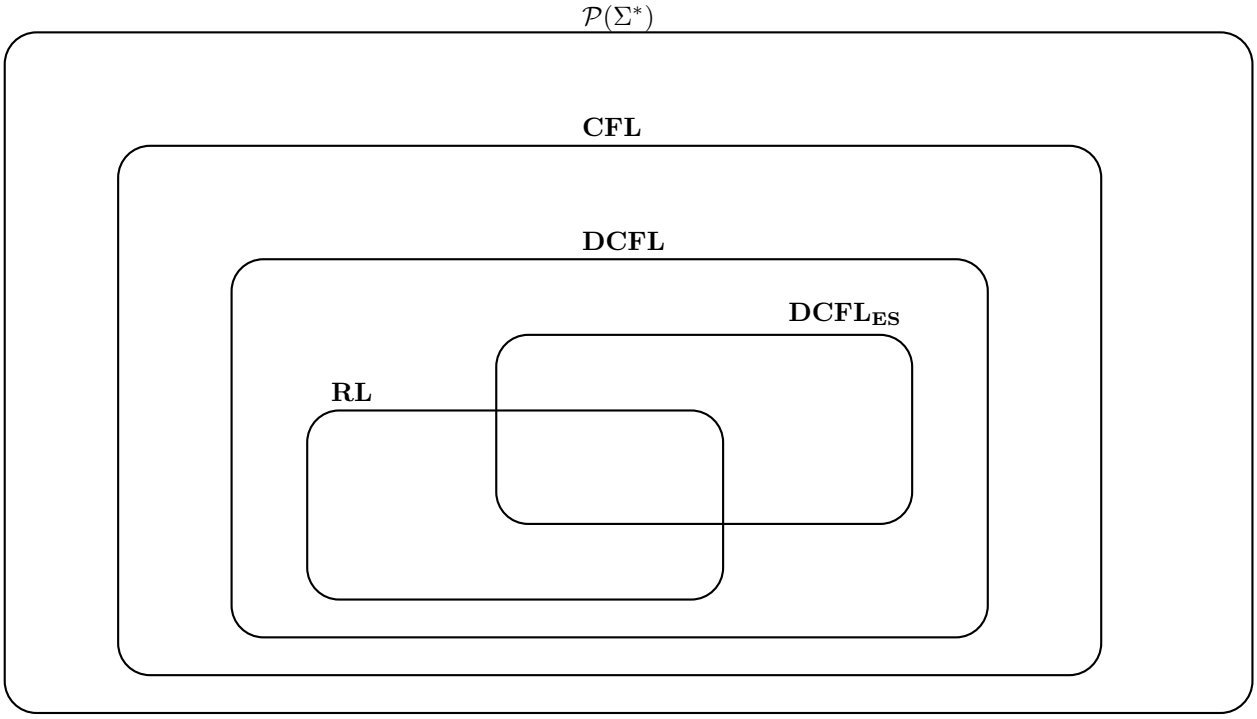
1. 15 points Consider the following languages:

   1. $L_1 = \{\mathtt{a}^n\mathtt{b} \mid n \geq 1\}$
   2. $L_2 = \{\mathtt{a}\mathtt{b}^n \mid n \geq 1\}$
   3. $L_3 = \{\mathtt{a}^n\mathtt{b}^m \mid n, m \geq 0\}$
   4. $L_4 = \{\mathtt{a}^n\mathtt{b}^{2n} \mid n \geq 1\}$
   5. $L_5 = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i + k = j\}$
   6. $L_6 = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 1\}$
   7. $L_7 = \{ww \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$
   8. $L_8 = \{w\mathtt{c}w^R \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$
   9. $L_9 = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w = w^R\}$
   10. $L_{10} = \{w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \neq xx \text{ for any } x \in \{\mathtt{a}, \mathtt{b}\}^*\}$

   (a) 10 points [★☆☆] Consider the following classes of languages.

   - **CFL**: the class of **context-free languages**.
   - **DCFL**: the class of **deterministic context-free languages**.
   - **DCFL$_{\text{ES}}$**: the class of **deterministic context-free languages by empty stacks**.
   - **RL**: the class of **regular languages**.

   The following Venn diagram shows the relationships between these language classes. Place the above languages in the following Venn diagram using their numbers (e.g., $1, 2, \ldots$).



   (b) 5 points [★☆☆] Some languages are **NOT DCFLs** but can still be defined by an **unambiguous** CFG. Find such a language in the **above list** and construct an **unambiguous CFG** for the language.

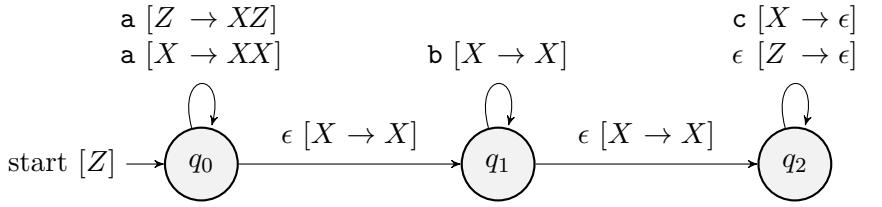
2. 10 points [★☆☆] **True/False questions.** Answer O for True and X for False.

1. The class of CFLs is closed under homomorphisms. ☐
2. The class of CFLs is closed under intersection. ☐
3. The class of DCFLs is closed under complement. ☐
4. The number of PDAs is countably infinite. ☐
5. The cardinality of the set of all languages $\mathcal{P}(\Sigma^*)$ over a finite set Σ of symbols is $\aleph_0$. ☐
6. We can prove a language is context-free using pumping lemma for CFLs. ☐
7. All NP-hard problems are NP problems. ☐
8. There exists an undecidable language that is recognized by a Turing machine. ☐
9. All NP-complete problems are polynomial-time reducible to each other. ☐
10. If P = NP, then all NP problems are decidable in polynomial time. ☐

3. 10 points Explain the languages defined by the **empty stacks** (L_E) or **final states** (L_F) of PDAs P_i.

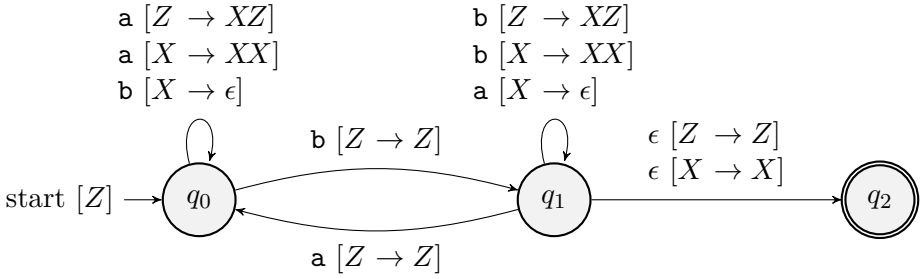
(a) 5 points [★☆☆] $L_E(P_1) = \{$ $\}$

$P_1 =$



(b) 5 points [★★☆] $L_F(P_2) = \{$ $\}$

$P_2 =$



4. 15 points We can freely **convert** a PDA to an **equivalent** CFG and vice versa.

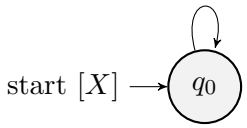
   (a) 5 points [☆☆☆] Consider the following CFG:

   $$S \to \epsilon \mid \texttt{a}S\texttt{a} \mid \texttt{b}S\texttt{b}$$

   Construct a PDA accepting the language of the above CFG by its **empty stacks** with a **single state**.

   (b) 5 points [★☆☆] Consider the following PDA P that accepts a language by its **empty stack**:

   $$P = \begin{array}{l} \texttt{a}\ [X \to YXY] \\ \texttt{b}\ [X \to XX] \\ \texttt{c}\ [Y \to \epsilon] \\ \epsilon\ [X \to \epsilon] \\ \epsilon\ [Y \to YY] \end{array}$$

   Fill in the blanks in the production rules of a CFG that represents the language accepted by the **empty stack** of the PDA, where each variable is a **stack symbol** satisfying the following:

   $$X \Rightarrow^* w \qquad \text{if and only if} \qquad (q_0, w, X) \vdash^* (q_0, \epsilon, \epsilon)$$

   $X \to$ ____

   $Y \to$ ____

   (c) 5 points [★★☆] In general, explain an **algorithm** that converts a PDA $P = (\{q_0\}, \Sigma, \Gamma, \delta, q_0, Z, \varnothing)$ with a **single state** accepting by **empty stack** into an equivalent CFG $G = (V, \Sigma, S, R)$.

5. 15 points Please construct new CFGs (G'_0, G'_1, and G'_2) from the given CFG (G_0, G_1, and G_2) by applying the following transformations.

   (a) 5 points [☆☆☆] Construct a CFG G'_0 consisting of productions produced by replacing **nullable variables** with ϵ in all combinations and removing all ϵ-productions in production rules in G_0.

$$G_0 = \begin{cases} S \to AAB \\ A \to \texttt{a}A \mid \epsilon \\ B \to \texttt{b}B \mid \texttt{b} \end{cases}$$

$G'_0 =$

   (b) 5 points [☆☆☆] Construct a CFG G'_1 by removing all **unit productions** and adding all possible **non-unit productions** of Y to X for each **unit pair** (Y, X) in G_1.

$$G_1 = \begin{cases} S \to A \mid \texttt{a}B \\ A \to \texttt{b}A \mid B \\ B \to \texttt{b} \end{cases}$$

$G'_1 =$

   (c) 5 points [★☆☆] Construct a CFG G'_2 by removing all productions that contain **non-generating variables** or come from **unreachable variables** in G_2.

$$G_2 = \begin{cases} S \to AB \mid CC \\ A \to \texttt{a}D \mid \texttt{a} \\ B \to \texttt{b}E \\ C \to \texttt{a}C\texttt{a} \mid \texttt{bb} \\ D \to \texttt{a}A \\ E \to \texttt{b}E \end{cases}$$

$G'_2 =$

6. 10 points [★★☆] Fill in the blanks in the **proof** showing that the language L is **not** a **context-free language (CFL)** using the **pumping lemma** for **CFLs**.

$$L = \{\mathtt{a}^p \mid p \text{ is a prime number}\}$$

1. Assume that any positive integer n is given. (i.e., $n \geq 1$)
2. Pick a word $L \ni z =$ [ ].
3. $|z| =$ [ ] $\geq n$.
4. Assume that any split $z = uvwxy$ satisfying ① $|vx| > 0$ and ② $|vwx| \leq n$ is given.
5. We need to show that ¬③ $uv^iwx^iy \notin L$ for some $i \geq 0$:

[ ]

7. 10 points [★★☆] Draw a transition diagram of a **Turing machine** that **computes** the following function:

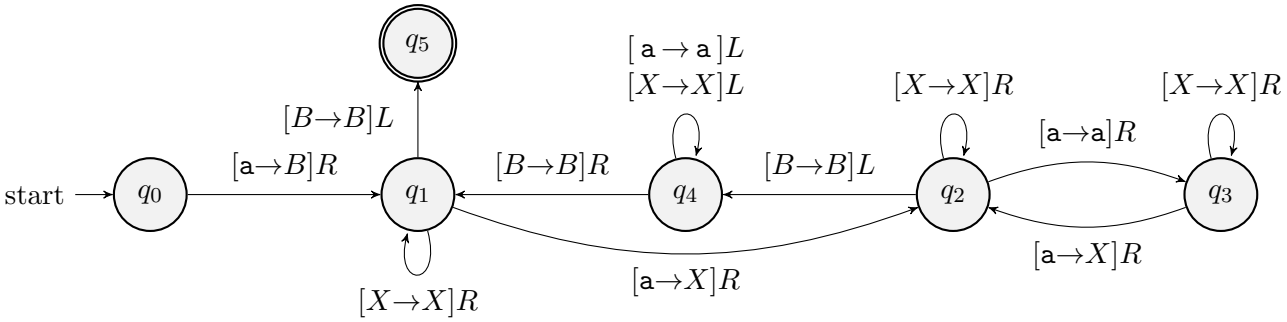
$$f : \{\mathtt{\#1}^n \mid n \geq 0\} \to \{\mathtt{0}, \mathtt{1}, \mathtt{\#}\}^* \text{ s.t. } f(\mathtt{\#1}^n) = (\text{the binary representation of } n)$$

For example, $f(\mathtt{\#}) = \epsilon$, $f(\mathtt{\#1}) = \mathtt{1}$, $f(\mathtt{\#11}) = \mathtt{10}$, $f(\mathtt{\#111}) = \mathtt{11}$, $f(\mathtt{\#1}^5) = \mathtt{101}$, and $f(\mathtt{\#1}^{13}) = \mathtt{1101}$.

[ ]

8. [8 points] [★★★] Write the **time complexity** of the following **Turing machine** M in a big-O notation in terms of $n = |w|$ for any input $w \in \{\texttt{a}\}^*$ and **explain** the reasoning behind your answer.

$M =$



Time complexity of $M =$

9. [7 points] [★★★] **Prove** or **disprove** the following statement:

The **recursively enumerable language (REL)** is closed under **complement**.

(Hint: The **universal language** $L_u = \{(M, w) \mid w \in L(M)$ where M is a TM and w is a binary word$\}$)

This is the last page.
I hope that your tests went well!