Lecture 1 – Mathematical Preliminaries
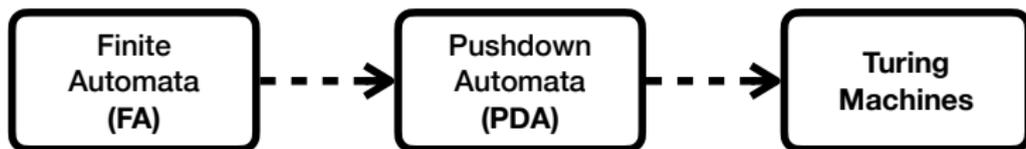## COSE215: Theory of Computation

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PLRG

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Roadmap: Towards Turing Machine

A Turing machine is a specific kind of **automaton**.

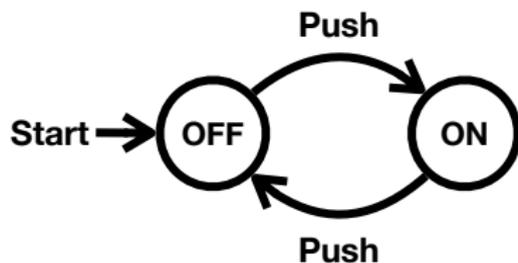- **Part 1: Finite Automata (FA)**
    - Regular Expressions (REs)
    - Regular Languages (RLs)
    - Applications: text search, etc.
- **Part 2: Pushdown Automata (PDA)**
    - Context-Free Grammars (CFGs)
    - Context-Free Languages (CFLs)
    - Applications: programming languages, natural language processing, etc.
- **Part 3: Turing Machines (TMs)**
    - Lambda Calculus (LC)
    - Recursively Enumerable Languages (RELs)
    - Undecidability and Intractability

Recall

Start → **OFF** ⟷ **ON**

Push

Push

Theorem

The current state is OFF *if and only if the button is pushed* **even** *times.*

- Is it possible to prove it?

    Let's learn **mathematical background and notation.**

Contents

Contents

Notation	Description
A, B	arbitrary **statements**.
$P(x)$	a **predicate** is a statement having variables (e.g., x).
$A \wedge B$	the **conjunction** of A and B. (i.e., "A and B").
$A \vee B$	the **disjunction** of A and B. (i.e., "A or B").
$\neg A$	the **negation** of A. (i.e., "not A").

$$(\textbf{De Morgan's Laws}) = \begin{cases} \neg(A \wedge B) = \neg A \vee \neg B \\ \neg(A \vee B) = \neg A \wedge \neg B \end{cases}$$

$$(\textbf{Truth Table}) = \begin{array}{cc||c|c||c|c|c} A & B & A \wedge B & \neg(A \wedge B) & \neg A & \neg B & \neg A \vee \neg B \\ \hline T & T & T & F & F & F & F \\ T & F & F & T & F & T & T \\ F & T & F & T & T & F & T \\ F & F & F & T & T & T & T \end{array}$$

Notations in Logics

Notation	Description
$A \Rightarrow B$	the **implication** of A and B (i.e., "if A then B" or "A implies B") (i.e., $\neg A \vee B$).
$A \Leftrightarrow B$	A **if and only if (iff)** B (i.e., $A \Rightarrow B \wedge B \Rightarrow A$).
$\forall x \in X. \, P(x)$	the **universal quantifier** (i.e., "for all x in X, $P(x)$ holds").
$\exists x \in X. \, P(x)$	the **existential quantifier** (i.e., "there exists x in X such that $P(x)$ holds").

$$(\textbf{Truth Table}) = \begin{array}{cc||c||cc|c} A & B & A \Rightarrow B & \neg A & B & \neg A \vee B \\ \hline T & T & T & F & T & T \\ T & F & F & F & F & F \\ F & T & T & T & T & T \\ F & F & T & T & F & T \end{array}$$

Notations in Set Theory

- A **set** is a collection of elements. For example,
    - (Integers) $= \mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$
    - (Natural Numbers) $= \mathbb{N} = \{0, 1, 2, \cdots\}$
    - (Squares of $\mathbb{N}$) $= \{x^2 \mid x \in \mathbb{N}\} = \{0, 1, 4, 9, 16, 25, 36, \cdots\}$
    - (Even Numbers) $= \{x \in \mathbb{N} \mid x \equiv 0 \pmod 2\} = \{0, 2, 4, 6, 8, \cdots\}$

    $$where \qquad a \equiv b \pmod n \iff \exists k \in \mathbb{Z}.\ a = b + kn$$

    (i.e., a is **congruent** to b **modulo** n)

- The **empty set** is denoted by $\varnothing$.
- The **cardinality** of a set X is denoted by $|X|$.
- x is an **element** of a set X is denoted by $x \in X$.
- X is a **subset** of Y is denoted by $X \subseteq Y$.

$$X \subseteq Y \iff \forall x \in X.\ x \in Y$$

- X is a **proper subset** of Y is denoted by $X \subset Y$.

$$X \subset Y \iff X \subseteq Y \wedge X \neq Y$$

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- The **union** of sets

$$X \cup Y = \{x \mid x \in X \vee x \in Y\}$$
$$\bigcup \mathcal{C} = X_1 \cup X_2 \cup \cdots \cup X_n = \{x \mid \exists X \in \mathcal{C}. \, x \in X\}$$

where $\mathcal{C} = \{X_1, X_2, \cdots, X_n\}$.

- The **intersection** of sets

$$X \cap Y = \{x \mid x \in X \wedge x \in Y\}$$
$$\bigcap \mathcal{C} = X_1 \cap X_2 \cap \cdots \cap X_n = \{x \mid \forall X \in \mathcal{C}. \, x \in X\}$$

where $\mathcal{C} = \{X_1, X_2, \cdots, X_n\}$.

- The **difference** of sets

$$X \setminus Y = \{x \mid x \in X \wedge x \notin Y\}$$

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- The **complement** of a set X is denoted by $\overline{X}$.

$$\overline{X} = \{x \mid x \in U \wedge x \notin X\}$$

where U is the **universal set**.

- The **power set** of a set X is denoted by 2^X or $\mathcal{P}(X)$.

$$2^X = \mathcal{P}(X) = \{Y \mid Y \subseteq X\}$$

- The **Cartesian product** of sets X and Y is denoted by $X \times Y$.

$$X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$$
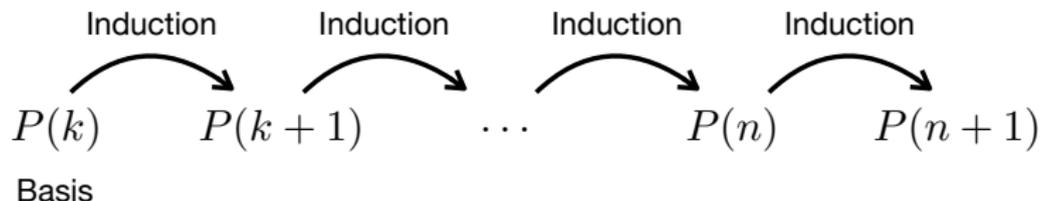
Contents

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Definition (Inductions on Integers)

Let $P(n)$ be a predicate on integers, and if

- **(Basis Case)** $P(k)$ holds where k is an integer, and
- **(Induction Case)** for all integer $n \geq k$, $P(n) \Rightarrow P(n+1)$,

then $P(i)$ holds for all $i \geq k$.

$P(n)$ is called **induction hypothesis** (I.H.).

$$
\overbrace{}^{\text{Induction}} \overbrace{}^{\text{Induction}} \overbrace{}^{\text{Induction}} \overbrace{}^{\text{Induction}}
$$

$P(k) \qquad P(k+1) \qquad \cdots \qquad P(n) \qquad P(n+1)$

Basis

Inductions on Integers – Example 1

Example

Prove that $\forall n \geq 0.\ \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Proof)

- (Basis Case): $0 = 0(0+1)/2$ $\quad\square$
- (Induction Case): Assume that it holds for n (I.H.)

  Then, let's prove it for $n + 1$:

$$\sum_{i=0}^{n+1} i = (n+1) + \sum_{i=0}^{n} i$$
$$= (n+1) + \frac{n(n+1)}{2} \qquad (\because I.H.)$$
$$= \frac{(n+1)(n+2)}{2} \quad\square$$

Example

Prove that $\forall n \geq 0.\ \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof)

- (Basis Case): $0^2 = 0(0+1)(2*0+1)/6$ □

- (Induction Case): Assume that it holds for n (I.H.).

  Then, let's prove it for $n+1$:

$$\sum_{i=0}^{n+1} i^2 = (n+1)^2 + \sum_{i=0}^{n} i^2$$

$$= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \qquad (\because I.H.)$$

$$= \frac{(n+1)(n+2)(2(n+1)+1)}{6} \quad \square$$

Structural Inductions – Inductive Definitions

In CS, we often define somethings as **inductively-defined sets**.
For example, we can define **trees** as follows:

Example (Inductive Definition of Trees)

A **tree** is defined as follows:

- **(Basis Case)** A single **node** N is a tree.

- **(Induction Case)** If $T_1, \cdots, T_n$ are trees, then a graph defined with a new root node N and edges from N to $T_1, \cdots, T_n$ is a tree.

Structural Inductions – Inductive Definitions
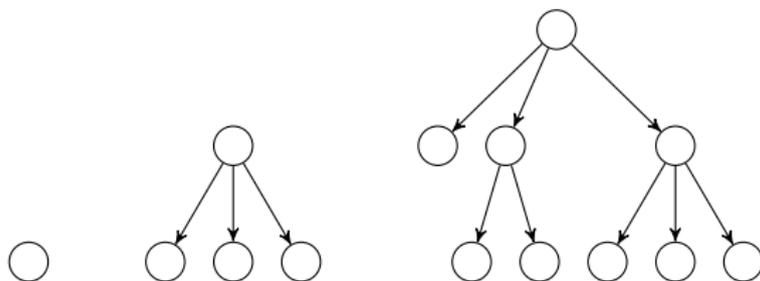
Another example is a set of **arithmetic expressions**:

Example (Inductive Definition of Arithmetic Expressions)

An **arithmetic expression** is defined as follows:

- **(Basis Case)** A **number** or a **variable** is an arithmetic expression.
- **(Induction Case)** If E and F are arithmetic expressions, then so are $E+F$, $E*F$, and (E).

```
    42                x              x + y

  42 * x            (x)         (x * y) * z

(2 + x) * y    x * (x * y)      (((((x))))
```

Structural Inductions

Definition (Structural Inductions)

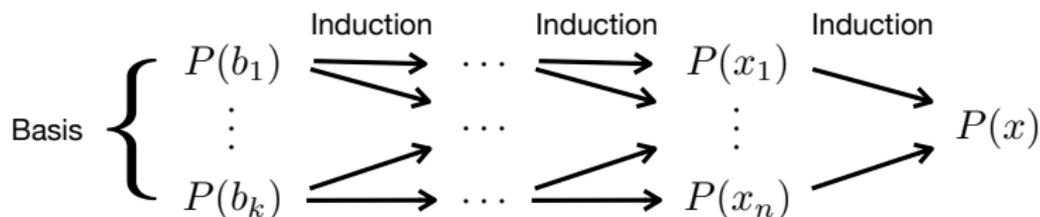Let $P(x)$ be a predicate on a **inductively-defined set** X, and if

- **(Basis Case)** $P(b_1), \cdots, P(b_k)$ hold for all basis cases $b_1, \cdots, b_k$.
- **(Induction Case)** for all $x \in X$,

$$P(x_1) \wedge \cdots \wedge P(x_n) \Rightarrow P(x)$$

where $x_1, \cdots, x_n$ are the **sub-structures** of x.

then $P(x)$ holds for all $x \in X$.

$P(x_1), \cdots, P(x_n)$ are called **induction hypotheses**.

Structural Inductions – Example 1

Example

Prove that for all tree T, the number of nodes in T is equal to the number of edges in T plus one.

Proof) Let $N(T)$ be the number of node and $E(T)$ be the number of edges in T. Let's prove $\forall T.\ N(T) = E(T) + 1$.

- (Basis Case): $N(T) = 1$ and $E(T) = 0$. $\quad\square$
- (Induction Case): Assume that it holds for $T_1, \cdots, T_n$ (I.H.). Then,

$$N(T) = 1 + \sum_{i=1}^{n} N(T_i)$$

$$= 1 + \sum_{i=1}^{n} (E(T_i) + 1) \qquad (\because I.H.)$$

$$= 1 + n + \sum_{i=1}^{n} E(T_i)$$

$$= 1 + E(T) \quad\square$$

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Example

Prove that for all arithmetic expression E, the number of left parentheses in E is equal to the number of right parentheses in E.

Proof) Let $L(E)$ be the number of left parentheses and $R(E)$ be the number of right parentheses in E. Let's prove $\forall E. \ L(E) = R(E)$.

- (Basis Case): $L(E) = R(E) = 0$ for numbers and variables. $\square$
- (Induction Case): Assume that it holds for E and F (I.H.). Then,

$$L(E\text{+}F) = L(E) + L(F) = R(E) + R(F) \qquad (\because \textit{I.H.})$$
$$= R(E\text{+}F) \quad \square$$
$$L(E\text{*}F) = L(E) + L(F) = R(E) + R(F) \qquad (\because \textit{I.H.})$$
$$= R(E\text{*}F) \quad \square$$
$$L((E)) = L(E) + 1 = R(E) + 1 \qquad (\because \textit{I.H.})$$
$$= R((E)) \quad \square$$

Mutual Inductions

Sometimes, we need to prove multiple predicates simultaneously.
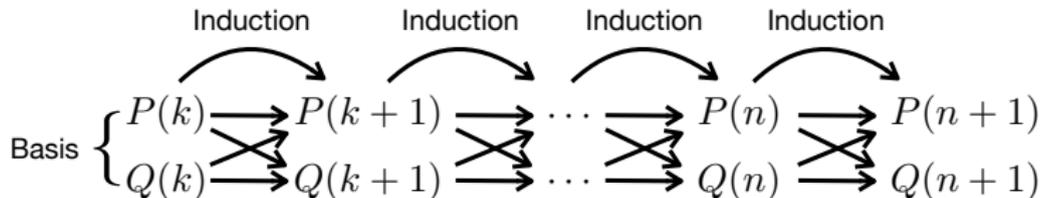
Definition (Mutual Inductions)

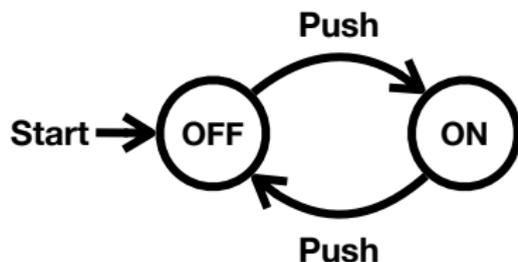Let $P(x)$ and $Q(x)$ are predicates on integers, and if

- **(Basis Case)** $P(k)$ and $Q(k)$ hold where k is an integer, and
- **(Induction Case)** for all $n \geq k$,

$$P(n) \wedge Q(n) \Rightarrow P(n+1) \wedge Q(n+1)$$

then $P(i)$ and $Q(i)$ hold for all $i \geq k$.

$P(n)$ and $Q(n)$ are called **induction hypotheses**.

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Theorem

The current state is OFF *if and only if the button is pushed* **even** *times.*

It is difficult to prove it with only one predicate.

Let's prove it with two predicates:

Theorem

The current state is OFF *if and only if the button is pushed* **even** *times, and the current state is* ON *if and only if the button is pushed* **odd** *times.*

Mutual Inductions – Example

Theorem

The current state is OFF *if and only if the button is pushed* **even** *times, and the current state is* ON *if and only if the button is pushed* **odd** *times.*

Proof) Let $S(i)$ be the current state after i times of pushing. Let's prove

$$\forall i \geq 0.\ S(i) = \text{OFF} \iff i \equiv 0 \ (\text{mod } 2) \qquad (P)$$

$$\forall i \geq 0.\ S(i) = \text{ON} \iff i \equiv 1 \ (\text{mod } 2) \qquad (Q)$$

- (Basis Case): Known facts: $S(0) = \text{OFF}$ and $0 \equiv 0 \ (\text{mod } 2)$

  - $(P, \Rightarrow)$: $S(0) = \text{OFF} \Rightarrow 0 \equiv 0 \ (\text{mod } 2)$ because $0 \equiv 0 \ (\text{mod } 2)$

  - $(P, \Leftarrow)$: $S(0) = \text{OFF} \Leftarrow 0 \equiv 0 \ (\text{mod } 2)$ because $S(0) = \text{OFF}$

  - $(Q, \Rightarrow)$: $S(0) = \text{ON} \Rightarrow 0 \equiv 1 \ (\text{mod } 2)$ because $S(0) \neq \text{ON}$

  - $(Q, \Leftarrow)$: $S(0) = \text{ON} \Leftarrow 0 \equiv 1 \ (\text{mod } 2)$ because $0 \not\equiv 1 \ (\text{mod } 2)$

Mutual Inductions – Example

- (Induction Case): Assume that it holds for n (I.H.):

$$S(n) = \text{OFF} \iff n \equiv 0 \pmod 2 \qquad (P - I.H.)$$
$$S(n) = \text{ON} \iff n \equiv 1 \pmod 2 \qquad (Q - I.H.)$$

- $(P, \Leftrightarrow)$:

$$
\begin{aligned}
S(n+1) = \text{OFF} &\iff S(n) = \text{ON} \\
&\iff n \equiv 1 \pmod 2 \qquad (\because Q - I.H.) \\
&\iff n + 1 \equiv 0 \pmod 2
\end{aligned}
$$

- $(Q, \Leftrightarrow)$:

$$
\begin{aligned}
S(n+1) = \text{ON} &\iff S(n) = \text{OFF} \\
&\iff n \equiv 0 \pmod 2 \qquad (\because P - I.H.) \\
&\iff n + 1 \equiv 1 \pmod 2
\end{aligned}
$$

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Symbols & Words

- We first define a finite and non-empty set of **symbols** Σ.
- A **word** $w \in \Sigma^*$ is a sequence of symbols.
  - $\Sigma = \{0, 1\}$ – binary symbols.

  $$\epsilon, 0, 1, 00, 01, 10010, \cdots \in \Sigma^*$$

  - $\Sigma = \{a, b, \cdots, z\}$ – lowercase letters.

  $$\epsilon, a, b, abc, hello, cs, students, \cdots \in \Sigma^*$$

  - $\Sigma = \{a \mid a \text{ is an Unicode character}\}$ – Unicode characters.

  $$\epsilon, 안녕하세요, こんにちは, \bigstar \blacksquare \blacktriangle \oplus, \cdots \in \Sigma^*$$

Symbols & Words

Notation	Description
ϵ	the **empty word**.
$w_1 w_2$	the **concatenation** of w_1 and w_2.
	(w_1 is a **prefix** of $w_1 w_2$ and w_2 is a **suffix** of $w_1 w_2$)
w^R	the **reverse** of w.
$|w|$	the **length** of w.
Σ^k	the set of all words of length k.
Σ^*	the set of all words (the **Kleene star**).
	(i.e., $\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \cdots = \bigcup_{k>0} \Sigma^k$)
Σ^+	the set of all words except ϵ (the **Kleene plus**).
	(i.e., $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \cdots = \bigcup_{k \geq 1} \Sigma^k$)

Languages

A **language** $L \subseteq \Sigma^*$ is a specific set of words.

When $\Sigma = \{0, 1\}$, we can define the following languages:

- $L = \{\epsilon, 0, 1\}$ – the empty word, zero, and one.

- $L = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \cdots\}$ – all binary words.

- $L = \{0^n 1^n \mid n \geq 0\}$ – equal number of consecutive zeros and ones.

- $L = \{10, 11, 101, 111, 1011, \cdots\}$ – ???

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- The **union**, **intersection**, and **difference** of languages:

$$L_1 \cup L_2 \qquad L_1 \cap L_2 \qquad L_1 \setminus L_2$$

- The **reverse** of a language:

$$L^R = \{ w^R \mid w \in L \}$$

- The **complement** of a language:

$$\overline{L} = \Sigma^* \setminus L$$

- The **concatenation** of languages:

$$L_1 L_2 = \{ w_1 w_2 \mid w_1 \in L_1 \wedge w_2 \in L_2 \}$$

Languages – Operations

- The **power** of a language defined inductively:

$$L^0 = \{\epsilon\}$$
$$L^n = L^{n-1}L \qquad (n \geq 1)$$

- The **Kleene star** of a language:

$$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots = \bigcup_{n \geq 0} L^n$$

- The **Kleene plus** of a language:

$$L^+ = L^1 \cup L^2 \cup L^3 \cup \cdots = \bigcup_{n \geq 1} L^n$$

- For any language L, are the following true?

  - $\epsilon \in L^*$ – Yes. Because $\epsilon \in L^0 \subseteq L^*$

  - $\epsilon \notin L^+$ – No. If $\epsilon \in L$, then $\epsilon \in L = L^1 \subseteq L^+$

Summary

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1. Mathematical Notations
    Notations in Logics
    Notations in Set Theory

2. Inductive Proofs
    Inductions on Integers
    Structural Inductions
    Mutual Inductions

3. Notations in Languages
    Symbols & Words
    Languages

- Basic Introduction of Scala

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