

Lecture 13 – Parse Trees and Ambiguity

COSE215: Theory of Computation

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2026 Spring

- A **context-free grammar (CFG)**:

$$G = (V, \Sigma, S, R)$$

- The **language** of a CFG G :

$$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$$

- A language L is a **context-free language (CFL)**:

$$\exists \text{ CFG } G. L(G) = L$$

- For a given word $w \in L(G)$, a **derivation** for w is $S \Rightarrow^* w$
- A sequence $\alpha \in (V \cup \Sigma)^*$ is a **sentential form** if $S \Rightarrow^* \alpha$.

1. Parse Trees

- Definition

- Yields

- Relationship between Parse Trees and Derivations

2. Ambiguity

- Ambiguous Grammars

- Eliminating Ambiguity

- Inherent Ambiguity

1. Parse Trees

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- Yields

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- Inherent Ambiguity

Consider the following CFG for balanced parentheses:

$$S \rightarrow \epsilon \mid (S) \mid SS$$

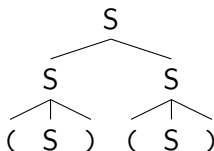
There are two different derivations for the sentential form $(S)(S)$:

$$\textcircled{1} \quad S \Rightarrow_L SS \Rightarrow_L (S)S \Rightarrow (S)(S)$$

$$\textcircled{2} \quad S \Rightarrow_R SS \Rightarrow_R S(S) \Rightarrow (S)(S)$$

However, **parse trees** focus on the structure of the derivations instead of considering the order of the derivation steps.

For example, the above two derivations have the same parse tree:



Definition (Parse Trees)

For a given CFG $G = (V, \Sigma, S, R)$, **parse trees** are trees satisfying:

- 1 The **root node** is labeled with the **start variable** S .
- 2 Each **internal node** is labeled with a **variable** $A \in V$.
If its children are labeled with:

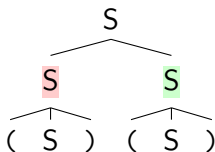
$$X_1, X_2, \dots, X_k$$

from the left to the right, then $A \rightarrow X_1 X_2 \dots X_k \in R$.

- 3 Each **leaf node** is labeled with a variable, symbol, or ϵ . However, if a leaf node is labeled with ϵ , it must be the only child of its parent.

$$S \rightarrow \epsilon \mid (S) \mid SS$$

A parse tree for $(S)(S)$:

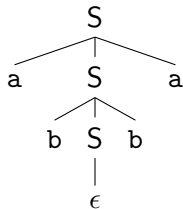


$$\textcircled{1} \quad S \Rightarrow_L \text{S S} \Rightarrow_L (S)S \Rightarrow (S)(S)$$

$$\textcircled{2} \quad S \Rightarrow_R \text{S S} \Rightarrow_R S(S) \Rightarrow (S)(S)$$

$$S \rightarrow \epsilon \mid aSa \mid bSb$$

A parse tree for abba:



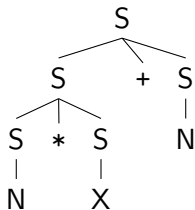
$$S \rightarrow N \mid X \mid S+S \mid S*S \mid (S)$$

$$N \rightarrow D \mid DN$$

$$D \rightarrow 0 \mid \dots \mid 9$$

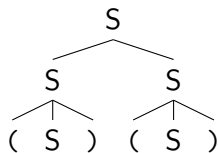
$$X \rightarrow a \mid \dots \mid z$$

A parse tree for $N*X+N$:

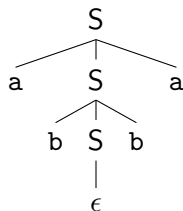


Definition (Yields)

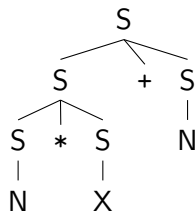
The sequence obtained by concatenating the labels (without ϵ) of the leaf nodes of a parse tree from left to right is called the **yield** of the parse tree.



Its yield is $(S)(S)$.



Its yield is $abba$.



Its yield is $N*X+N$.

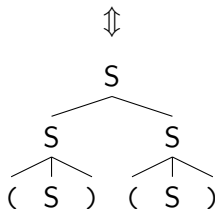
Theorem (Parse Trees and Derivations)

For a given CFG $G = (V, \Sigma, S, R)$, for any sequence $\alpha \in (V \cup \Sigma)^*$:

$$S \Rightarrow^* \alpha \iff \exists \text{ parse tree } T. \text{ s.t. } T \text{ yields } \alpha$$

For example, consider the sequence $(S)(S)$:

$$S \Rightarrow SS \Rightarrow (S)S \Rightarrow (S)(S)$$



1. Parse Trees

Definition

Yields

Relationship between Parse Trees and Derivations

2. Ambiguity

Ambiguous Grammars

Eliminating Ambiguity

Inherent Ambiguity

Is there always a **unique** parse tree for a given sentential form?

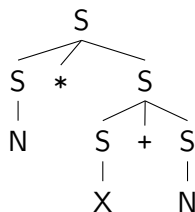
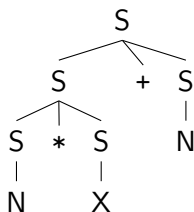
$$S \rightarrow N \mid X \mid S+S \mid S*S \mid (S)$$

$$N \rightarrow D \mid DN$$

$$D \rightarrow 0 \mid \dots \mid 9$$

$$X \rightarrow a \mid \dots \mid z$$

For example, consider the sentential form $N*X+N$:



Actually, there are **two** parse trees for $N*X+N$.

Definition (Ambiguous Grammar)

A context-free grammar $G = (V, \Sigma, S, R)$ is **ambiguous** if there exist two distinct parse trees for a word $w \in \Sigma^*$. If not, G is **unambiguous**.

Theorem

Let $G = (V, \Sigma, S, R)$ be a CFG. Then, the following numbers are equal for any sequence of variables or symbols $w \in (V \cup \Sigma)^*$:

- 1 The number of parse trees whose yields are w .
- 2 The number of left-most derivations for w .
- 3 The number of right-most derivations for w .

Proof) We can convert a left-most (or right-most) derivation for a word w into the corresponding parse tree for w and vice versa.

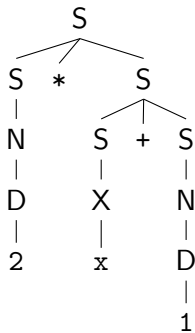
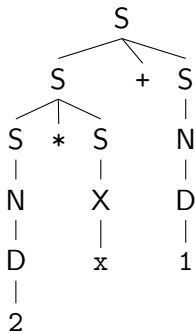
$$S \rightarrow N \mid X \mid S+S \mid S*S \mid (S)$$

$$N \rightarrow D \mid DN$$

$$D \rightarrow 0 \mid \dots \mid 9$$

$$X \rightarrow a \mid \dots \mid z$$

This grammar is **ambiguous** because there are **two** parse trees for the word $2 * x + 1$:



So, there are **two** left-most (or right-most) derivations for $2 * x + 1$.

$$\begin{aligned}
 S &\rightarrow N \mid X \mid S+S \mid S*S \mid (S) \\
 N &\rightarrow D \mid DN \\
 D &\rightarrow 0 \mid \dots \mid 9 \\
 X &\rightarrow a \mid \dots \mid z
 \end{aligned}$$

There are **two** left-most derivations for $2 * x + 1$:

- 1 Applying the production rule $S \rightarrow S+S$ first:

$$\begin{aligned}
 S &\Rightarrow_L S+S &\Rightarrow_L S*S+S &\Rightarrow_L N*S+S &\Rightarrow_L D*S+S &\Rightarrow_L 2*S+S \\
 &\Rightarrow_L 2*X+S &\Rightarrow_L 2*x+S &\Rightarrow_L 2*x+N &\Rightarrow_L 2*x+D &\Rightarrow_L 2*x+1
 \end{aligned}$$

- 2 Applying the production rule $S \rightarrow S*S$ first:

$$\begin{aligned}
 S &\Rightarrow_L S*S &\Rightarrow_L N*S &\Rightarrow_L D*S &\Rightarrow_L 2*S &\Rightarrow_L 2*S+S \\
 &\Rightarrow_L 2*X+S &\Rightarrow_L 2*x+S &\Rightarrow_L 2*x+N &\Rightarrow_L 2*x+D &\Rightarrow_L 2*x+1
 \end{aligned}$$

Unfortunately,

- There is **NO** general algorithm to remove ambiguity from a CFG.
- There is even **NO** algorithm to determine a CFG is ambiguous.

Fortunately, there are well-known techniques to manually **eliminate** the ambiguity in a given grammar commonly used in programming languages.

$$S \rightarrow N \mid X \mid S+S \mid S*S \mid (S)$$

$$N \rightarrow D \mid DN$$

$$D \rightarrow 0 \mid \dots \mid 9$$

$$X \rightarrow a \mid \dots \mid z$$

For example, an equivalent but unambiguous grammar is:

$$S \rightarrow T \mid S+T$$

$$T \rightarrow F \mid T*F$$

$$F \rightarrow N \mid X \mid (S)$$

$$N \rightarrow D \mid DN$$

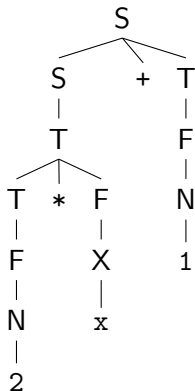
$$D \rightarrow 0 \mid \dots \mid 9$$

$$X \rightarrow a \mid \dots \mid z$$

Eliminating Ambiguity

Now, the unique parse tree for $2 * x + 1$ is:

$$\begin{aligned} S &\rightarrow T \mid S+T \\ T &\rightarrow F \mid T * F \\ F &\rightarrow N \mid X \mid (S) \\ N &\rightarrow D \mid DN \\ D &\rightarrow 0 \mid \dots \mid 9 \\ X &\rightarrow a \mid \dots \mid z \end{aligned}$$



Let's try to understand how to eliminate the ambiguity in the original grammar.

First, analyze why the original grammar is ambiguous.

$$S \rightarrow N \mid X \mid S+S \mid S*S \mid (S)$$

$$N \rightarrow D \mid DN$$

$$D \rightarrow 0 \mid \dots \mid 9$$

$$X \rightarrow a \mid \dots \mid z$$

- **Precedence** is not specified between different operators (+ and *).

- For example, two parse trees for $1 * 2 + 3$ interpreted as:

$$1 * (2 + 3) \quad \text{and} \quad (1 * 2) + 3$$

- Let's give * higher precedence than + to interpret it as $(1 * 2) + 3$.

- **Associativity** for the same operator (+ or *).

- For example, two parse trees for $1 + 2 + 3$ interpreted as:

$$1 + (2 + 3) \quad \text{and} \quad (1 + 2) + 3$$

- Let's give the left-associativity to + to interpret it as $(1 + 2) + 3$.

To enforce the **precedence**, define new variables F for factors and T for terms:

- A **factor** is a number, a variable, or a parenthesized expression:

$$42, \quad x, \quad (1 + 2), \quad \dots$$

In the grammar, F is defined as:

$$F \rightarrow N \mid X \mid (S)$$

- A **term** is the multiplication of one or more factors:

$$42, \quad 2 * x, \quad 2 * (1 + 2), \quad 1 * (x * y) * z, \quad \dots$$

In the grammar, T is defined as:

$$T \rightarrow F \mid T * F$$

- An **expression** is the addition of one or more terms:

$$42, \quad 1 + 2, \quad 1 + 2 * 3, \quad (1 + 2) * 3 + 4, \quad \dots$$

In the grammar, S is defined as:

$$S \rightarrow T \mid S + T$$

The unambiguous grammar is:

$$S \rightarrow T \mid S+T$$

$$T \rightarrow F \mid T*F$$

$$F \rightarrow N \mid X \mid (S)$$

$$N \rightarrow D \mid DN$$

$$D \rightarrow 0 \mid \dots \mid 9$$

$$X \rightarrow a \mid \dots \mid z$$

- This grammar supports the **left-associativity** of + and *. Why?

The unambiguous grammar is:

$$\begin{aligned} S &\rightarrow T \mid S+T \\ T &\rightarrow F \mid T*F \\ F &\rightarrow N \mid X \mid (S) \\ N &\rightarrow 0 \mid \dots \mid 9 \mid 0N \mid \dots \mid 9N \\ X &\rightarrow a \mid \dots \mid z \end{aligned}$$

- This grammar supports the **left-associativity** of + and *. Why?
 - $S \rightarrow S+T$ and $T \rightarrow T*F$ are **left-recursive**.
- Then, how to support the **right-associativity** of + and *?
 - Replace the **left-recursive** rules with **right-recursive** rules!

$$\begin{aligned} S &\rightarrow T \mid T+S \\ T &\rightarrow F \mid F*T \\ \dots \end{aligned}$$

So far, we have discussed the **ambiguity** for **grammars**.
We will now discuss the **inherent ambiguity** for **languages**.

Definition (Inherent Ambiguity)

A language L is **inherently ambiguous** if all CFGs whose languages are L are ambiguous. (i.e. there is no unambiguous grammar for L)

For example, the following language is **inherently ambiguous**:

$$L = \{a^i b^j c^k \mid i, j, k \geq 0 \wedge (i = j \vee j = k)\}$$

An example of ambiguous grammar for L is:

$$\begin{aligned} S &\rightarrow L \mid R & L &\rightarrow X \mid Lc & R &\rightarrow Y \mid aR \\ & & X &\rightarrow \epsilon \mid aXb & Y &\rightarrow \epsilon \mid bYc \end{aligned}$$

While we can prove that L is inherently ambiguous using the Ogden's lemma¹, we will not discuss it in this course.

¹https://en.wikipedia.org/wiki/Ogden's_lemma

Summary

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Definition

Yields

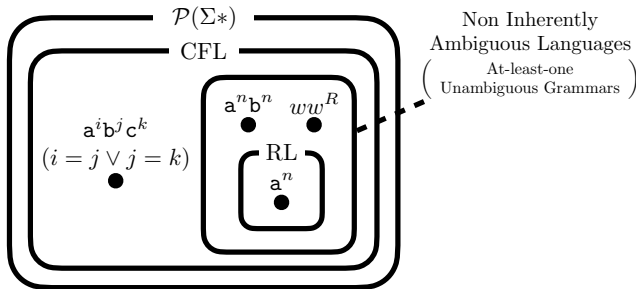
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- The midterm exam will be given in class.
- **Date:** 13:30-14:45 (1 hour 15 minutes), April 22 (Wed.).
- **Location:**
 - **IT-Education Hall B102:** Students with Student ID 2025XXXX
 - **IT-Education Hall 611:** All other students
- **Coverage:** Lectures 1 – 13 (No coding question)
- **Format:** 7–9 questions with closed book and closed notes
- Please refer to the **previous exams** in the course website:

<https://plrg.korea.ac.kr/courses/cose215/>

- **Apr. 27 (Mon.):** No class
- **Apr. 29 (Wed.):** Recorded video lecture (Online).

- Pushdown Automata (PDA)

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