

Lecture 17 – Deterministic Pushdown Automata (DPDA)

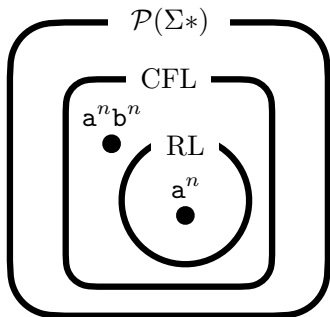
COSE215: Theory of Computation

Jihyeok Park



2026 Spring

- A **pushdown automaton (PDA)** is an extension of ϵ -NFA with a **stack**. Thus, PDA is **non-deterministic**.
 - Acceptance by **final states**
 - Acceptance by **empty stacks**
- Then, how about **deterministic PDA (DPDA)**?
- What is the **language class** of DPDA? Still, CFL?



PDA_{FS}
(by final states)

||

PDA_{ES}
(by empty stacks)

||

CFG

1. Deterministic Pushdown Automata (DPDA)

2. Deterministic Context-Free Languages (DCFLs)

Fact 1: $\text{DCFL} \subsetneq \text{CFL}$

Fact 2: $\text{RL} \subsetneq \text{DCFL}$

3. Languages Accepted by Empty Stacks of DPDA (DCFL_{ES})

Fact 3: $\text{DCFL}_{\text{ES}} \subsetneq \text{DCFL}$

Fact 4: $\text{DCFL}_{\text{ES}} = \text{DCFL}$ having Prefix Property

Fact 5: $\text{RL} \not\subseteq \text{DCFL}_{\text{ES}}$

4. Inherent Ambiguity of DCFLs

Fact 6: $\text{DCFL} \subsetneq \text{Non Inherently Ambiguous Languages}$

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We can check it with the following conditions:

- 1 $|\delta(q, a, X)| \leq 1$ for all $q \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$.
- 2 If $\delta(q, \epsilon, X) \neq \emptyset$, then $\delta(q, a, X) = \emptyset$ for all $a \in \Sigma$.

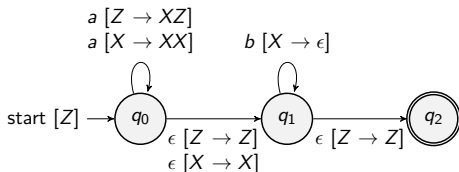
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For example, is the following PDA deterministic?



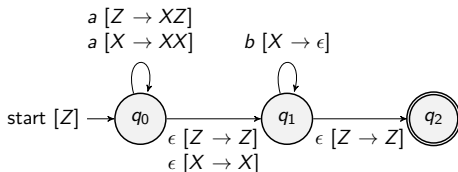
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No, because it has multiple transitions for (q_0, ab, Z) .

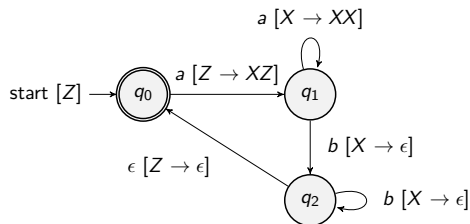
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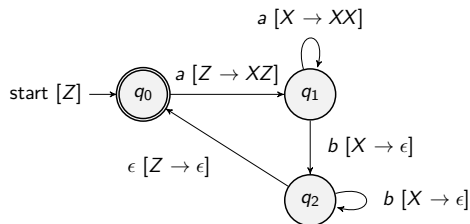
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However, the following PDA is **deterministic**:



$$\begin{aligned}
 (q_0, aabb, Z) &\vdash (q_1, abb, XZ) \\
 &\vdash (q_1, bb, XXZ) \\
 &\vdash (q_2, b, XZ) \\
 &\vdash (q_2, \epsilon, Z) \\
 &\vdash (q_0, \epsilon, \epsilon)
 \end{aligned}$$

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Definition (Deterministic Context-Free Languages (DCFLs))

A language L is a **deterministic context-free language (DCFL)** if and only if there exists a DPDA P such that $L = L_F(P)$ where $L_F(P)$ is the language accepted by **final states** of P .

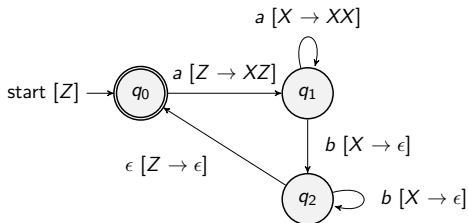
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For example, the following language is a DCFL:

$$L = \{a^n b^n \mid n \geq 0\}$$

because it is accepted by **final states** of the following **DPDA**:



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The following language is a CFL but not a DCFL:

$$L = \{ww^R \mid w \in \{a, b\}^*\} \in \text{CFL} \setminus \text{DCFL}$$

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The formal proof is complex, but we can intuitively understand it with the following two example words in L :

- $ww^R = abba \in L$ where $w = ab$
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When we read b after ab , we need to consider two possible actions:

- ① pop Y for b (for $abba$) or
- ② push Y for b (for $abbbbba$).

Fact 2: $RL \subsetneq DCFL$

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Then, we can construct a DPDA P that accepts L as follows:

$$P = (Q, \Sigma, \{Z\}, \delta_P, q_0, Z, F)$$

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$$(q_0, w, Z) \vdash^* (q, \epsilon, Z) \iff \delta^*(q_0, w) = q \quad \square$$

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② $DCFL \setminus RL \neq \emptyset$: We already know that the following language is a DCFL but not an RL:

$$L = \{a^n b^n \mid n \geq 0\} \in DCFL \setminus RL$$

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Definition ($DCFL_{ES}$)

A language L is a **deterministic context-free language by empty stacks ($DCFL_{ES}$)** if and only if there exists a DPDA P such that $L = L_E(P)$ where $L_E(P)$ is the language accepted by **empty stacks** of P .

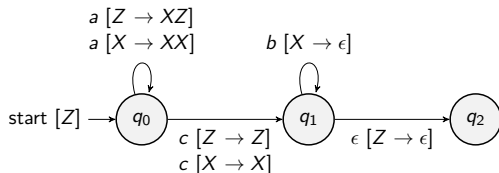
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For example, the following language is a DCFL_{ES}:

$$L = \{a^n cb^n \mid n \geq 0\}$$

because it is accepted by empty stacks of the following DPDA:



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① All DCFL_{ES} s are DCFLs **BUT** ② there is a DCFL but not a DCFL_{ES} .

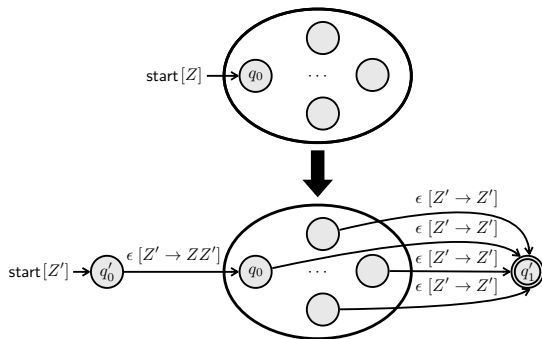
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We can generalize it as **prefix property** of DCFL_{ES} .

Definition (Prefix Property)

A language L has the **prefix property** if and only if for any word $w \in L$, any proper prefix of w is not in L :

$$\forall x, y \in \Sigma^*. ((xy \in L \wedge y \neq \epsilon) \implies x \notin L)$$

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For example, the following language is a **DCFL** but does **NOT** have the **prefix property** because $\epsilon \in L$ is a proper prefix of

$$L = \{a^n b^n \mid n \geq 0\}$$

Thus, L is a **DCFL** but **NOT** a DCFL_{ES} .

Fact 5: $RL \not\subseteq DCFL_{ES}$

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- $RL \setminus DCFL_{ES} \neq \emptyset$: For example, the following language is a **RL** but does **NOT** have the **prefix property**:

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because $aa \in L$ is a proper prefix of $aaaa \in L$.

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It satisfies the following fact:

$$\text{DCFL} \subsetneq \text{Non Inherently Ambiguous Languages}$$

We prove this fact by the following three steps:

- ① $\text{DCFL}_{ES} \subseteq \text{Non Inherently Ambiguous Languages}$
- ② $\text{DCFL} \subseteq \text{Non Inherently Ambiguous Languages}$ (using ①)
- ③ $\text{Non Inherently Ambiguous Languages} \setminus \text{DCFL} \neq \emptyset$

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- ① A DCFL_{ES} has an unambiguous grammar: For a given DCFL_{ES} L and its corresponding DPDA P , we can define a CFG for P as follows:

- For all $0 \leq j < n$,

$$S \rightarrow A_{0,j}^Z$$

- For all transition $(q_j, X_1 \cdots X_m) \in \delta(q_i, a, X)$ where $q_i \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$ and combinations $0 \leq k_1, \dots, k_m < n$:

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For any word $w \in L$, w has a unique moves from the initial configuration to the final configuration in P . And, we know that:

$$A_{i,j}^X \Rightarrow^* w \quad \text{if and only if} \quad (q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$$

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- ① A DCFL_{ES} has an unambiguous grammar: For a given DCFL_{ES} L and its corresponding DPDA P , we can define a CFG for P as follows:

- For all $0 \leq j < n$,

$$S \rightarrow A_{0,j}^Z$$

- For all transition $(q_j, X_1 \cdots X_m) \in \delta(q_i, a, X)$ where $q_i \in Q$, $a \in \Sigma \cup \{\epsilon\}$, and $X \in \Gamma$ and combinations $0 \leq k_1, \dots, k_m < n$:

$$A_{i,k_m}^X \rightarrow a A_{j,k_1}^{X_1} A_{k_1,k_2}^{X_2} \cdots A_{k_{m-1},k_m}^{X_m}$$

For any word $w \in L$, w has a unique moves from the initial configuration to the final configuration in P . And, we know that:

$$A_{i,j}^X \Rightarrow^* w \quad \text{if and only if} \quad (q_i, w, X) \vdash^* (q_j, \epsilon, \epsilon)$$

Since the leftmost derivation of w simulates the unique moves of w in P , there is only one leftmost derivation for w . Thus, CFG is **unambiguous**.

Fact 6: DCFL \subsetneq Non Inherently Ambiguous Languages

All DCFLs have at least one corresponding unambiguous grammars **BUT** there exists a non inherently ambiguous language that is not a DCFL.

- ② A DCFL has an unambiguous grammar: For a given DCFL L , we can define another DCFL L' with a special symbol $\$$ as follows:

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For example, $L = \{a^n b^n \mid n \geq 0\}$ is DCFL, then $L' = \{a^n b^n \$ \mid n \geq 0\}$ is a DCFL_{ES} and its **unambiguous grammar** G' is:

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- ③ Non Inherently Ambiguous Languages \setminus DCFL $\neq \emptyset$: The following language is a **non inherently ambiguous language** but **not** a **DCFL**:

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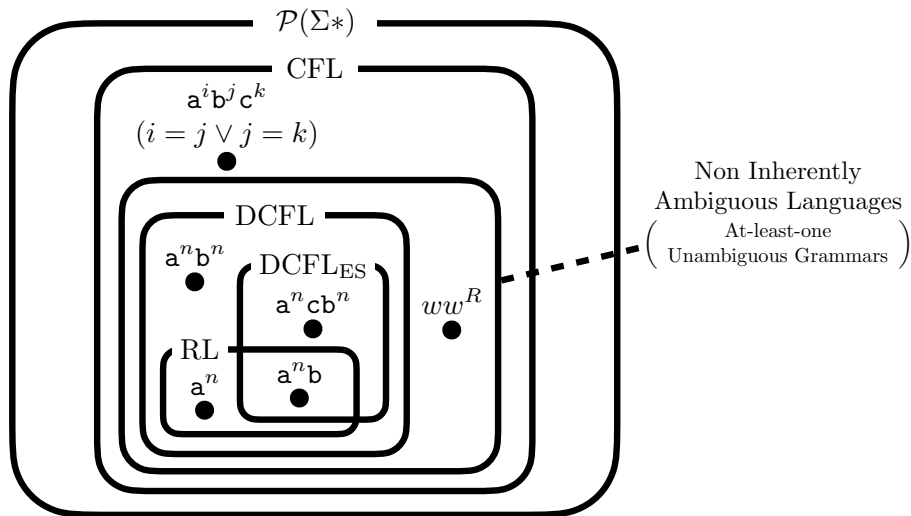
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because the following **unambiguous grammar** G represents L :

$$S \rightarrow aSa \mid bSb \mid \epsilon$$

but we already know that L is **not** a **DCFL**.



- Please see this document on GitHub:

<https://github.com/ku-plrg-classroom/docs/tree/main/cose215/pda-examples>

- The due date is 23:59 on May 18 (Mon.).
- Please only submit `Implementation.scala` file to [LMS](#).

- Normal Forms of Context-Free Grammars

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