

# Lecture 18 – Normal Forms of Context-Free Grammars

## COSE215: Theory of Computation

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2026 Spring

- A **context-free grammar (CFG)** is a 4-tuple:

$$G = (V, \Sigma, S, R)$$

where

- $V$ : a finite set of **variables** (nonterminals)
  - $\Sigma$ : a finite set of **symbols** (terminals)
  - $S \in V$ : the **start variable**
  - $R \subseteq V \times (V \cup \Sigma)^*$ : a set of **production rules**.
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- How to **simplify** a CFG?

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- How to **simplify** a CFG?

Let's put it in **Chomsky normal form (CNF)**!

1. Chomsky Normal Form (CNF)
2. Eliminating  $\epsilon$ -Productions  
Nullable Variables
3. Eliminating Unit Productions  
Unit Pairs
4. Eliminating Useless Variables  
Generating Variables  
Reachable Variables
5. Putting CFG in CNF

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## Definition (Chomsky Normal Form)

A CFG  $G$  is in **Chomsky normal form (CNF)** if all productions are of the form for some  $A, B, C \in V$  and  $a \in \Sigma$ :

$$A \rightarrow BC \quad \text{OR} \quad A \rightarrow a \quad \text{OR} \quad S \rightarrow \epsilon$$

where  $B \neq S$  and  $C \neq S$ . And  $S \rightarrow \epsilon$  is allowed only if  $\epsilon \in L(G)$ .

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Consider the following CFG:

$$\begin{array}{l} S \rightarrow 0ABC \mid 1B \mid BB \quad A \rightarrow ABB0 \mid C \quad C \rightarrow CC \mid \epsilon \\ B \rightarrow 0B \mid 1 \quad D \rightarrow 1D \mid AA \end{array}$$

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Is it possible to put this CFG in CNF? **Yes!**

$$S \rightarrow XS_1 \mid XB \mid YB \mid BB \quad A \rightarrow AA_1 \mid BA_2 \quad B \rightarrow XB \mid 1$$

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Let's learn how to put a CFG in CNF!

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# Eliminating $\epsilon$ -Productions

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We can do it by following the steps below:

- 1 Find all **nullable variables**.
- 2 Construct a new CFG by **replacing** nullable variables with  $\epsilon$  in **all combinations** and **removing** all  $\epsilon$ -productions in production rules.

## Definition (Nullable Variables)

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We can inductively define the set of **nullable variables**:

- **(Basis Case)** If  $A \rightarrow \epsilon \in R$ , then  $A$  is nullable.
- **(Induction Case)** If  $A \rightarrow X_1 X_2 \cdots X_n \in R$  and  $X_1, X_2, \dots, X_n$  are all nullable, then  $A$  is nullable.

Consider the following CFG:

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- 1 Find all **unit pairs**.
- 2 Construct a new CFG by **adding** all possible non-unit productions of  $B$  to  $A$  for each unit pair  $(A, B)$ .

## Definition (Unit Pairs)

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We can inductively define the set of **unit pairs**:

- **(Basis Case)**  $(A, A)$  is a unit pair for all  $A \in V$ .
- **(Induction Case)** If  $(A, B)$  is a unit pair and  $B \rightarrow C \in R$ , then  $(A, C)$  is a unit pair.

# Eliminating Unit Productions – Example

After eliminating  $\epsilon$ -productions:

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What are useless variables?

- **Non-generating variables:** Variables that cannot derive any word.
- **Unreachable variables:** Variables unreachable from the start variable.

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Is it possible to eliminate **useless variables**?

Yes, we can do it by following the steps below:

- 1 Find all **generating variables**.
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## Definition (Generating Variables)

For a given CFG  $G = (V, \Sigma, S, R)$ , a variable  $A \in V$  is a **generating variable** if for some  $w \in \Sigma^*$ ,

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We can inductively define the set of **generating variables**:

- **(Basis Case)** There is no basis case.
- **(Induction Case)** If  $A \rightarrow \alpha \in R$  and  $\alpha$  contains only symbols or generating variables, then  $A$  is a generating variable.

## Definition (Reachable Variables)

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We can inductively define the set of **reachable variables**:

- **(Basis Case)** The start variable  $S$  is reachable variable.
- **(Induction Case)** If  $A \in V$  is a reachable variable and  $A \rightarrow \alpha \in R$ , then all variables in  $\alpha$  are reachable variables.

# Eliminating Useless Variables – Example

After eliminating  $\epsilon$ -productions and unit productions:

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- 1 Find all **generating variables**:  $\{S, A, B, D\}$  –  $C$  is non-generating.

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## Putting CFG in CNF

Our goal is to put a CFG in **Chomsky normal form (CNF)** consisting of:

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- 3 Rewrite all RHSs whose length  $> 1$  to contain only variables: if a symbol  $a$  appears in the RHS, replace it with a new variable  $A$  and introduce a new production rule  $A \rightarrow a$ .

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- 4 Replace all RHSs whose length is greater than 2 with a chain of variables. To do so, if  $A \rightarrow X_1X_2 \cdots X_n$  is a production with  $n > 2$ , then replace it with a sequence of productions:

$$A \rightarrow X_1A_1 \quad A_1 \rightarrow X_2A_2 \quad \cdots \quad A_{n-2} \rightarrow X_{n-1}X_n$$

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- 5 If  $\epsilon$  is in the original language, add a production  $S \rightarrow \epsilon$  (or  $S' \rightarrow \epsilon$ ).

# Putting CFG in CNF – Example 1

Let's put the following CFG in CNF:

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- ③ Rewrite all RHSs whose length  $> 1$  to contain only variables:

$$\begin{aligned} S &\rightarrow XAB \mid XB \mid YB \mid BB & X &\rightarrow 0 \\ A &\rightarrow ABBX \mid BBX & Y &\rightarrow 1 \\ B &\rightarrow XB \mid 1 \end{aligned}$$

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 \end{aligned}$$

- ③ Rewrite all RHSs whose length  $> 1$  to contain only variables:

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- ④ Replace all RHSs whose length  $> 2$  with a chain of variables:

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 S &\rightarrow XS_1 \mid XB \mid YB \mid BB & A &\rightarrow AA_1 \mid BA_2 & B &\rightarrow XB \mid 1 \\
 S_1 &\rightarrow AB & A_1 &\rightarrow BA_2 & X &\rightarrow 0 \\
 & & A_2 &\rightarrow BX & Y &\rightarrow 1
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- 5 If  $\epsilon$  is in the original language, add a production  $S \rightarrow \epsilon$ :

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$$S' \rightarrow \epsilon \mid AS_1 \mid AB \quad S \rightarrow AS_1 \mid AB \quad S_1 \rightarrow SB \quad A \rightarrow a \quad B \rightarrow b$$

1. Chomsky Normal Form (CNF)
2. Eliminating  $\epsilon$ -Productions  
Nullable Variables
3. Eliminating Unit Productions  
Unit Pairs
4. Eliminating Useless Variables  
Generating Variables  
Reachable Variables
5. Putting CFG in CNF

- Please see this document for the exercise.

<https://github.com/ku-plrg-classroom/docs/tree/main/cose215/cfg-cnf>

- Please implement the following functions in `Implementation.scala`.
- It is just an exercise, and you **don't need to submit** anything.

- Properties of Context-Free Languages

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