

Lecture 20 – The Pumping Lemma for Context-Free Languages

COSE215: Theory of Computation

Jihyeok Park



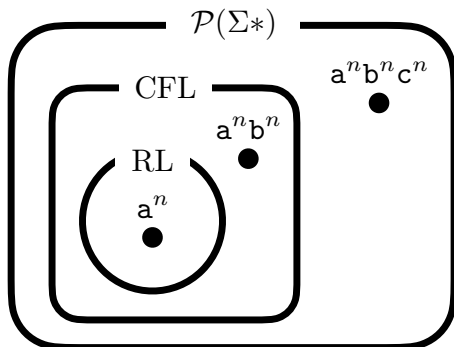
2026 Spring

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- We could use it to **prove** that some languages are **NOT** regular.
- Is there a similar lemma for **Context-Free Languages (CFLs)**?
- Yes. We can use it to prove the following language is **NOT** a CFL:

$$L = \{a^n b^n c^n \mid n \geq 0\}$$



1. Pumping Lemma for Context-Free Languages

Size of Parse Trees in Chomsky Normal Form

Pumping Lemma

Proof of Pumping Lemma

Application: Proving Languages are Not Context-Free

2. Examples

Example 1: $L = \{a^n b^n c^n \mid n \geq 0\}$

Example 2: $L = \{0^n 10^n 10^n \mid n \geq 0\}$

Example 3: $L = \{ww \mid w \in \{a, b\}^*\}$

Example 4: $L = \{a^i b^j c^j \mid i, j \geq 0 \wedge i \geq 2j\}$

Example 5: $L = \{w \in \{a, b, c\}^* \mid N_a(w) = N_b(w) = N_c(w)\}$

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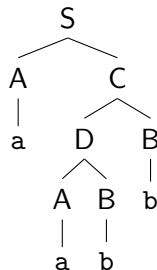
Theorem (Size of Parse Trees in Chomsky Normal Form)

For a given CFG G in Chomsky Normal Form, for all $w \in L(G)$, if the length of the longest path in the parse tree of w is n , then $|w| \leq 2^{n-1}$. Note that the length of a path is the number of edges in the path.

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For example, consider the following CFG in CNF, and the parse tree of $w = aabb$. The length of the longest path in the parse tree is 4, and the length of the word is 4. Thus, $|w| = 4 \leq 2^3 = 2^{n-1}$.

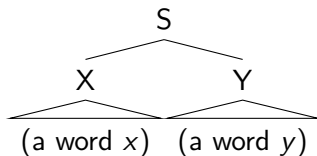
$$\begin{aligned} S &\rightarrow \epsilon \mid AC \mid AB \\ D &\rightarrow AC \mid AB \\ C &\rightarrow DB \\ A &\rightarrow a \\ B &\rightarrow b \end{aligned}$$


Proof) Let's perform induction on the length of the longest path n .

- **(Basis Case)** $n = 1$. Then, $|\epsilon| = 0 \leq 2^{1-1}$ and $|a| = 1 \leq 2^{1-1}$.



- **(Induction Case)** The first rule of S is in the form of $S \rightarrow XY$. The length of the longest path in the parse tree of X (or Y) is less than or equal to $n - 1$. If $X \Rightarrow^* x \in \Sigma^*$ and $Y \Rightarrow^* y \in \Sigma^*$, then $|x| \leq 2^{n-2}$ and $|y| \leq 2^{n-2}$ (\because I.H.). Thus, $|w| = |x| + |y| \leq 2^{n-2} + 2^{n-2} = 2^{n-1}$.



Lemma (Pumping Lemma for Context-Free Languages)

For a given CFL L , **there exists** a *positive integer* n such that **for all** $z \in L$, if $|z| \geq n$, **there exists** a split $z = uvwxy$ such that

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$A =$ L is context-free



$B = \exists n > 0. \forall z \in L. |z| \geq n \Rightarrow \exists z = uvwxy. \textcircled{1} \wedge \textcircled{2} \wedge \textcircled{3}$

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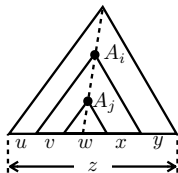
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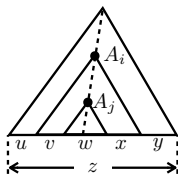
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- Split the word $z = uvwxy$ as follows:



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and

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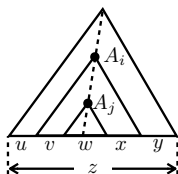


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- ① $|vx| > 0$
 - Since $i < j$, the word vw derived from A_i is not equal to the word w derived from A_j . ($\because S \rightarrow \epsilon$ never occurs in the middle of the parse tree.)
 - Thus, vx is not an empty word, and $|vx| > 0$.



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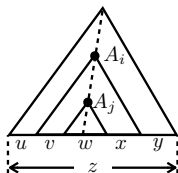
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• ② $|vwx| \leq n$

- Since $p - m \leq i$, the length of the longest path from A_i in the parse tree of z is $p - i + 1$ is less than or equal to $m + 1$.
- By Theorem of Size of Parse Trees in CNF, the length of the word vwx is less than or equal to $2^m = n$.

Proof of Pumping Lemma - ③



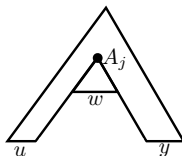
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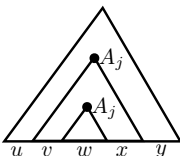
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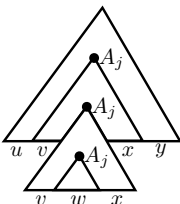
uwy
(uv^0wx^0y)



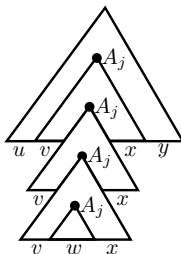
$uvwxy$
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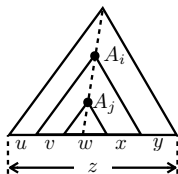
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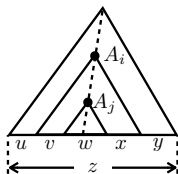


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Lemma (Pumping Lemma for Context-Free Languages)

$A = L \text{ is context-free}$



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$$\begin{aligned} \neg B &= \forall n > 0. \neg(\forall z \in L. |z| \geq n \Rightarrow \exists z = uvwxy. \textcircled{1} \wedge \textcircled{2} \wedge \textcircled{3}) \\ &= \forall n > 0. \exists z \in L. \neg(|z| \geq n \Rightarrow \exists z = uvwxy. \textcircled{1} \wedge \textcircled{2} \wedge \textcircled{3}) \\ &= \forall n > 0. \exists z \in L. |z| \geq n \wedge \neg(\exists z = uvwxy. \textcircled{1} \wedge \textcircled{2} \wedge \textcircled{3}) \\ &= \forall n > 0. \exists z \in L. |z| \geq n \wedge \forall z = uvwxy. \neg(\textcircled{1} \wedge \textcircled{2} \wedge \textcircled{3}) \\ &= \forall n > 0. \exists z \in L. |z| \geq n \wedge \forall z = uvwxy. \neg(\textcircled{1} \wedge \textcircled{2}) \vee \neg\textcircled{3} \\ &= \forall n > 0. \exists z \in L. |z| \geq n \wedge \forall z = uvwxy. (\textcircled{1} \wedge \textcircled{2}) \Rightarrow \neg\textcircled{3} \end{aligned}$$

To prove a language L is **NOT** context-free, we need to show that

$$\forall n > 0. \exists z \in L. |z| \geq n \wedge \forall z = uvwxy. ((1) \wedge (2)) \Rightarrow \neg(3)$$

- ① $|vx| > 0$
- ② $|vwx| \leq n$
- ③ $\forall i \geq 0. uv^iwx^iy \in L$

Note that $\neg(3) = \exists i \geq 0. uv^iwx^iy \notin L$.

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Note that $\neg \textcircled{3} = \exists i \geq 0. uv^i wx^i y \notin L$.

We can prove this by following the steps below:

- 1 Assume **any** positive integer n is given.
- 2 **Pick** a word $z \in L$.
- 3 Show that $|z| \geq n$.
- 4 Assume **any** split $z = uvwxy$ is given ($\textcircled{1} |vx| > 0 \wedge \textcircled{2} |vwx| \leq n$).
- 5 $\neg \textcircled{3}$ Pick $i \geq 0$, and show that $uv^i wx^i y \notin L$ using $\textcircled{1}$ and $\textcircled{2}$.

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- ⑤ Let $i = 0$. We need to show that \neg ③ $uv^0wx^0y \notin L$:
 - Since ② $|vwx| \leq n$,

$$vx = a^p b^q \quad (\text{or } vx = b^p c^q)$$

where $0 \leq p, q \leq n$.

- Since ① $|vx| > 0$, we can remove at least one a or b (or b or c) from z without changing the number of c's (or a's) when $i = 0$.
- It means that $uv^0wx^0y \notin L$. □

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Let's prove that L is **NOT** context-free using the Pumping Lemma:

$$L = \{0^n 10^n 10^n \mid n \geq 0\}$$

- ① Assume any positive integer n is given.
- ② Let $z = 0^n 10^n 10^n \in L$.
- ③ $|z| = n + 1 + n + 1 + n = 3n + 2 \geq n$.
- ④ Assume any split $z = uvwxy$ is given, and ① $|vx| > 0 \wedge$ ② $|vwx| \leq n$.
- ⑤ Let $i = 0$. We need to show that \neg ③ $uv^0wx^0y \notin L$:
 - Since ② $|vwx| \leq n$,

 vx cannot cover the third block (or the first block) consisting of 0's.
 - Since ① $|vx| > 0$, we can remove at least one 0 in the first or second blocks (or second or third blocks) from z without changing the number of 0's in the third block (or first block) when $i = 0$.
 - It means that $uv^0wx^0y \notin L$. □

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Let's prove that L is **NOT** context-free using the Pumping Lemma:

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 - Since ② $|vwx| \leq n$,
 vx cannot cover both two different blocks consisting of a's (or b's).
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Let's prove that L is **NOT** context-free using the Pumping Lemma:

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- 3 $|z| = 2n + n + 2n = 5n \geq n$.

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 - Therefore, $uv^2wx^2y \notin L$.
 - Otherwise,
 - vx covers only b. Thus, $vx = b^k$ and $k > 0$ (\because ① $k = |vx| > 0$).
 - $v^2wx^2y = a^{2n}b^{n+k}c^{2n} \notin L$ ($\because k > 0 \Rightarrow 2n < 2(n+k)$).

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Let's prove that L is **NOT** context-free:

$$L = \{w \in \{a, b, c\}^* \mid N_a(w) = N_b(w) = N_c(w)\}$$

where $N_a(w)$, $N_b(w)$, and $N_c(w)$ are the number of a's, b's, and c's in w .

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- Consider a regular expression $R = a^*b^*c^*$ and its language:

$$L(R) = \{a^i b^j c^k \mid i, j, k \geq 0\}$$

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- Since it is a contradiction, L is **NOT** context-free. □

1. Pumping Lemma for Context-Free Languages

Size of Parse Trees in Chomsky Normal Form

Pumping Lemma

Proof of Pumping Lemma

Application: Proving Languages are Not Context-Free

2. Examples

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- Turing Machines (TMs)

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