Lecture 22 – Examples of Turing Machines

COSE215: Theory of Computation

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PLRG

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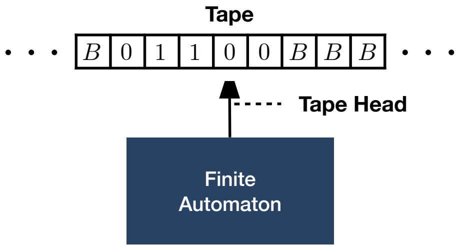
Recall



- A **Turing machine (TM)** is a **deterministic** FA with a **tape**.
  1. A **tape** is an infinite sequence of cells containing **tape symbols**. (The **blank symbol** B is a special symbol representing an empty cell.)
  2. A **tape head** points to the current cell.
  3. A **transition** performs the following operations depending on the current 1) **state** and 2) **tape symbol** pointed by the tape head:
     - **Change** the current **state**.
     - **Replace** the current **tape symbol** pointed by the tape head.
     - **Move** the **tape head** left or right.
- We can use Turing machines as **computing machines**.

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   - Example 3: $L = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^{i \times j} \mid i, j \geq 0\}$

2. Turing Machines as Computing Machines
   - Example 4: Flip Bits – $f(w \in \{0, 1\}^*) = (\text{flip of } w)$
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Example 1: $L = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^n b^n c^n \mid n \geq 0\}$$

$\cdots$	B	a	a	b	b	c	c	B	$\cdots$

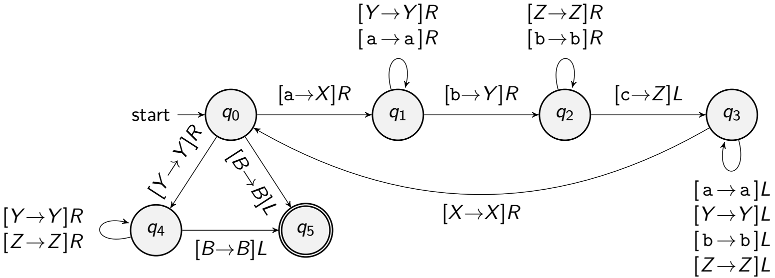
```
while there are a's do
    Find and Replace a with X
    Find and Replace b with Y
    Find and Replace c with Z
Check if only Y's and Z's are left
```

Example 1: $L = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^n b^n c^n \mid n \geq 0\}$$

See the example for aabbcc $\in L(M)$.[1]



[1] https://plrg.korea.ac.kr/courses/cose215/materials/tm-an-bn-cn.pdf

Example 2: $L = \{ww \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{ww \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$$

···	B	a	b	b	a	b	b	B	···

1: Find the middle of the input by repeatedly replacing leftmost and rightmost a's (or b's) with X's (or Y's)
2: Replace all X's (or Y's) with a's (or b's) in the first half
3: **while** there are input symbols in the first half **do**
4: Replace a (or b) with X (or Y) in the first half
5: Find and Replace matched X (or Y) with Z in the second half

Example 2: $L = \{ww \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$

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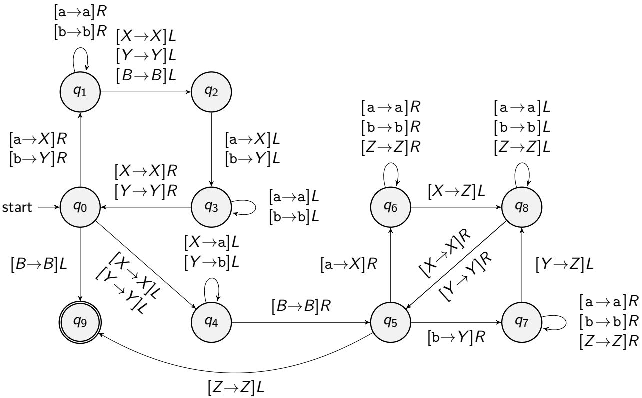
Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{ww \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$$

See the example for $\mathtt{abbabb} \in L(M)$.[2]



[2] https://plrg.korea.ac.kr/courses/cose215/materials/tm-w-w.pdf

Example 3: $L = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^{i \times j} \mid i, j \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^{i \times j} \mid i, j \geq 0\}$$

$\cdots$	B	a	a	b	b	b	c	c	c	c	c	c	B	$\cdots$

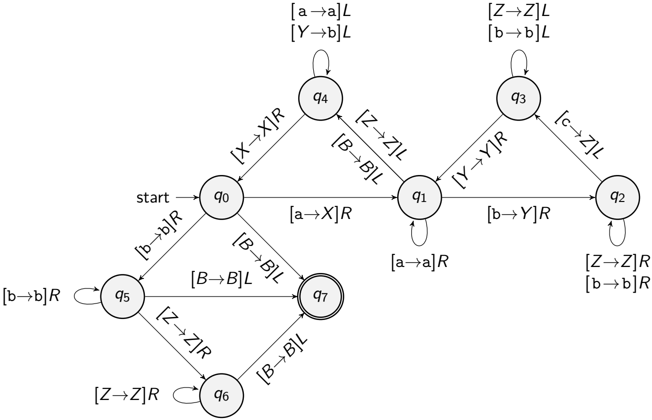
1: **while** there are a's **do**
2: Find and Replace a with X
3: **while** there are b's **do**
4: Find and Replace b with Y
5: Find and Replace c with Z
6: Roll back all Y's to b's
7: Check if only b's and Z's are left

Example 3: $L = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$

Construct a **Turing machine** that **accepts** the language:

$$L(M) = \{a^i b^j c^{i \times j} \mid i, j \geq 0\}$$

See the example for aabbbcccccc $\in L(M)$.[3]



[3]https://plrg.korea.ac.kr/courses/cose215/materials/tm-ai-bj-cij.pdf

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Example 4: Flip Bits – $f(w \in \{0,1\}^*) = (\text{flip of } w)$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0,1\}^*) = (\text{the flip of each bit in } w)$$

w =	$\cdots$	B	1	0	1	1	1	0	0	B	$\cdots$
$f(w)$ =	$\cdots$	B	0	1	0	0	0	1	1	B	$\cdots$

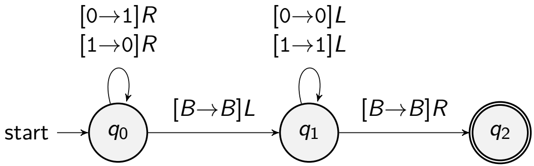
1: Flip each bit of the input: $1 \to 0$ and $0 \to 1$
2: Go to the first input symbol

Example 4: Flip Bits – $f(w \in \{0,1\}^*) = (\text{flip of } w)$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0,1\}^*) = (\text{the flip of each bit in } w)$$

See the example for $f(1011100) = 0100011$.[4]



[4] https://plrg.korea.ac.kr/courses/cose215/materials/tm-flip.pdf

Example 5: Unary Addition – $f(1^n{+}1^m) = 1^{n+m}$

Construct a **Turing machine** that **computes** the function:

$$f(1^n{+}1^m) = 1^{n+m} \qquad \text{where} \qquad n, m \geq 0$$

w =	$\cdots$	B	1	1	1	+	1	1	B	$\cdots$
$f(w)$ =	$\cdots$	B	1	1	1	1	1	B	B	$\cdots$

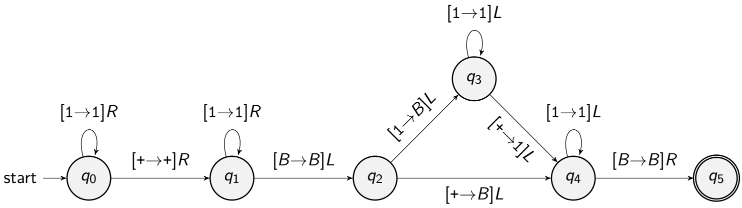
```
Find + after 1's
if the last symbol is 1 then
    Find and Remove the last 1
    Find and Replace the + with 1
else
    Remove the +
Go to the first input symbol
```

Example 5: Unary Addition – $f(1^n+1^m) = 1^{n+m}$

Construct a **Turing machine** that **computes** the function:

$$f(1^n+1^m) = 1^{n+m} \qquad \text{where} \qquad n, m \geq 0$$

See the example for $f(111+11) = 11111$.[5]



[5] https://plrg.korea.ac.kr/courses/cose215/materials/tm-unary-add.pdf

Example 6: Binary Increment – $f(w \in \{0,1\}^*) = w+1$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0,1\}^*) = w + 1 \qquad \text{where} \qquad w \text{ starts with } 1$$

$w =$	$\cdots$	B	1	0	1	0	1	1	1	1	B	$\cdots$
$f(w) =$	$\cdots$	B	1	0	1	1	0	0	0	0	B	$\cdots$

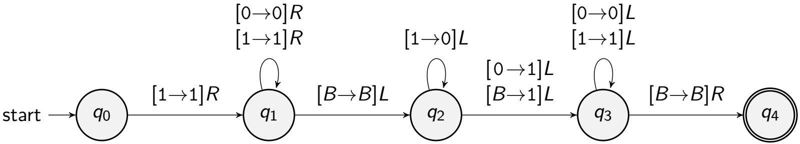
1: Check if the first bit is 1.
2: Move to the end of the input.
3: Repeatedly replace the rightmost 1 with 0.
4: Replace 0 (or B) with 1.
5: Go to the first input symbol.

Example 6: Binary Increment – $f(w \in \{0,1\}^*) = w+1$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{0,1\}^*) = w + 1 \qquad \text{where} \qquad w \text{ starts with } 1$$

See the example for $f(10101111) = 10110000$.[6]



[6] https://plrg.korea.ac.kr/courses/cose215/materials/tm-inc.pdf

Example 7: Data Copy – $f(w \in \{a, b\}^*) = ww$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{a, b\}^*) = ww$$

$w =$	$\cdots$	B	a	b	b	B	B	B	B	$\cdots$
$f(w) =$	$\cdots$	B	a	b	b	a	b	b	B	$\cdots$

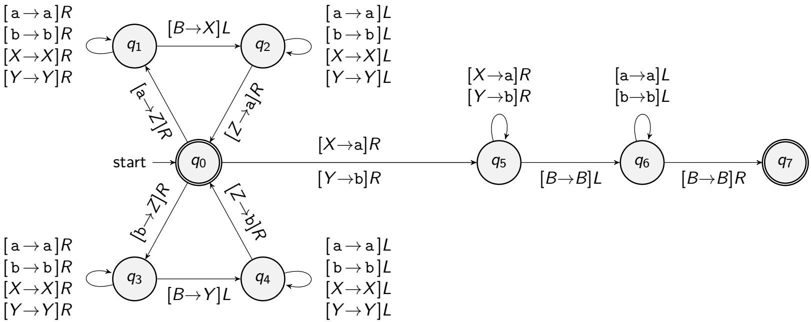
1: **while** there are input symbols **do**
2: Find and Replace a (or b) with Z
3: Find and Fill the first blank with X (or Y) for a (or b)
4: Roll back Z to the original a (or b)
5: Replace X's and Y's with a's and b's
6: Go to the first input symbol

Example 7: Data Copy – $f(w \in \{\mathtt{a}, \mathtt{b}\}^*) = ww$

Construct a **Turing machine** that **computes** the function:

$$f(w \in \{\mathtt{a}, \mathtt{b}\}^*) = ww$$

See the example for $f(\mathtt{abb}) = \mathtt{abbabb}$.[7]



[7] https://plrg.korea.ac.kr/courses/cose215/materials/tm-data-copy.pdf

Summary

1. Turing Machines as Word Recognizers

   Example 1: $L = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

   Example 2: $L = \{ww \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$

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Next Lecture

- Extensions of Turing Machines

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