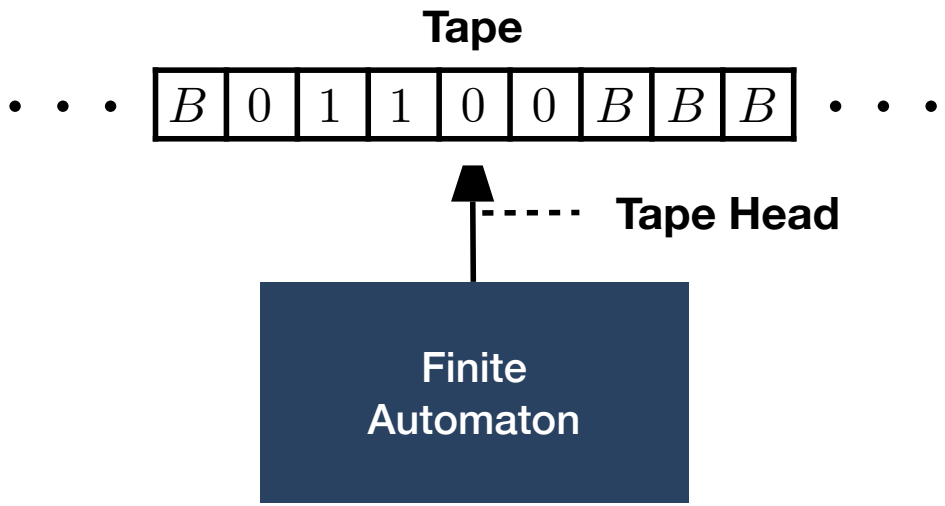
Lecture 24 – The Origin of Computer Science

COSE215: Theory of Computation

Jihyeok Park

2026 Spring

Recall



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- A standard **TM** is the **most powerful model of computation**.

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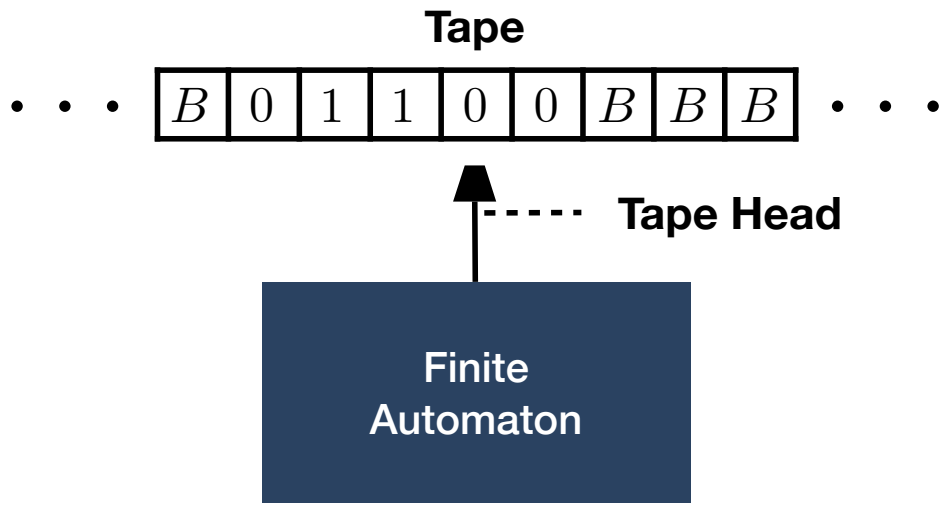

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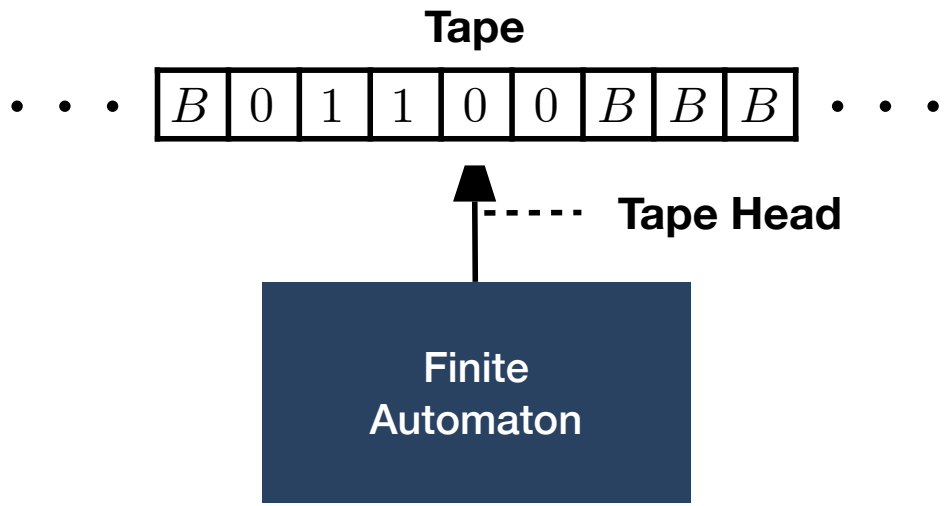

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- Why did **Alan Turing** invent the **TM**?
- Why is TM the **origin of Computer Science**?

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1. Gödel's Incompleteness Theorem
    - Example: Continuum Hypothesis
    - Gödel Numbering
2. Entscheidungsproblem – Decision Problem
    - Disproof using Turing Machine
    - Disproof using Lambda Calculus
3. Church-Turing Thesis

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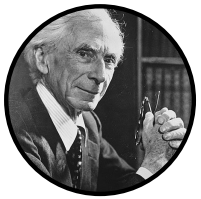
Gödel's Incompleteness Theorem

There is a fatal contradiction in our current set theory! How about the following statement? **True** or **False**?

$$\text{Let } R = \{x \mid x \notin x\}, \text{ then } R \in R?$$

Russell's Paradox



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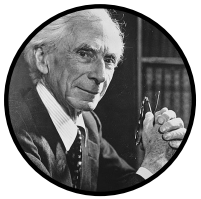

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We can always avoid such paradoxes by adding rigorous **axioms**!
(e.g., **ZF** - Zermelo-Fraenkel set theory)

I believe that we can build a mathematical system that

1. cannot **prove** and **disprove** the same statement (**Consistent**)
2. can **prove** every **True** statement (**Complete**)

Hilbert's Program

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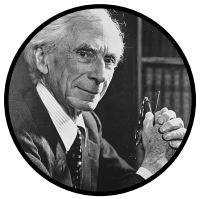

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Hilbert's Program

Unfortunately, I proved that any **Consistent** formal system containing **arithmetic** must be **Incomplete**.

Gödel's 1st Incompleteness Theorem

K. Gödel
(1906–1978)

Example: Continuum Hypothesis

- **Cardinality**: The number of elements in a set.

$$|\{3, 42, 7\}| = 3$$

Example: Continuum Hypothesis

PLRG

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- A set is **countably infinite** if there is a **bijection** between the set and the set of natural numbers (the cardinality of natural numbers is $\aleph_0$).
  - The set of **non-negative even numbers** is **countably infinite**.

$$\mathbb{N} \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \{n \in \mathbb{N} \mid n \geq 0 \land n \equiv 0 \pmod 2\} \text{ where } f(n) = 2n \text{ and } f^{-1}(n) = \frac{n}{2}$$

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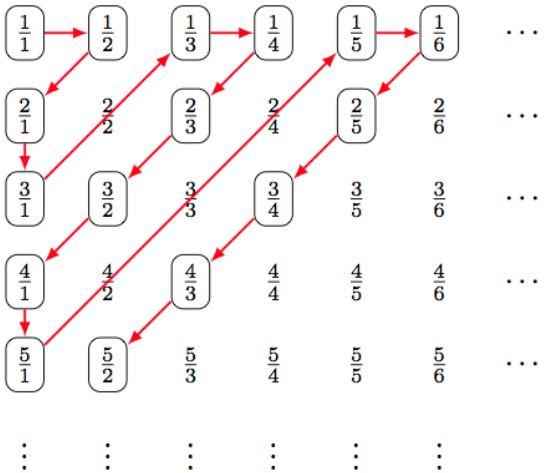
  - The set of **rational numbers** is **countably infinite**.



Example: Continuum Hypothesis

- A set of **real numbers** between 0 and 1 is **uncountably infinite** and its cardinality ($2^{\aleph_0}$) is strictly larger than the set of natural numbers ($2^{\aleph_0} > \aleph_0$) because of **Cantor's diagonal argument**:

n	$f(n)$												
1	0	.	**3**	1	4	1	5	9	2	6	5	3	...
2	0	.	3	**7**	3	7	3	7	3	7	3	7	...
3	0	.	1	4	**2**	8	5	7	1	4	2	8	...
4	0	.	7	0	7	**1**	0	6	7	8	1	1	...
5	0	.	3	7	5	0	**0**	0	0	0	0	0	...
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- Let $\aleph_1$ be the **smallest cardinal** greater than $\aleph_0$.
- **Continuum Hypothesis**:

$$\aleph_1 = 2^{\aleph_0} \quad \iff \quad \nexists \aleph.\ \aleph_0 < \aleph < 2^{\aleph_0}$$

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2	0 . 3 **7** 3 7 3 7 3 7 3 7 ...
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4	0 . 7 0 7 **1** 0 6 7 8 1 1 ...
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- Let $\aleph_1$ be the **smallest cardinal** greater than $\aleph_0$.
- **Continuum Hypothesis**:

$$\aleph_1 = 2^{\aleph_0} \quad \Longleftrightarrow \quad \nexists \aleph.\ \aleph_0 < \aleph < 2^{\aleph_0}$$

- K. Gödel and P. Cohen showed we **CANNOT** either prove or disprove the **Continuum Hypothesis** in **ZFC** (**ZF** + Axiom of **C**hoice).

Gödel Numbering

- **Gödel Numbering**: Assign a unique number to each symbol and string in a formal language.

Symbol	$\sim$	$\vee$	$\supset$	$\exists$	$=$	0	s	(	)	,	$+$
Number	1	2	3	4	5	6	7	8	9	10	11
Symbol	$\times$	x	y	z	p	q	r	P	Q	R	
Number	12	13	14	15	16	17	18	19	20	21	

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- We will use **prime numbers** to encode strings:

$$\Gamma(x_1 \cdots x_n) = \prod_{i=1}^{n} p_i^{x_i}$$

  where p_i is the i-th prime number.
- For example, $\Gamma(0{=}0) = 2^6 \times 3^5 \times 5^6 = 243,000,000$.

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- Gödel used this idea to encode **formulas** and **proofs** in **first-order arithmetic**, and then proved his famous **Incompleteness Theorem**.[1]

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Gödel's 1st Incompleteness Theorem

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Definition (Demonstration – **Dem**)

$$[x \ \textbf{Dem} \ y] \iff \Gamma^{-1}(x) \text{ is a proof of } \Gamma^{-1}(y)$$

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Let's define f and g (as defined in the previous slide, $\Gamma(x) = 13$):

$$f(x) = \neg\exists p.\ [p \textbf{ Dem } (x \textbf{ Sub } (13, x))] \qquad g = f(\Gamma(f))$$

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This is **true** but **not provable** in any sufficiently expressive consistent formal system (i.e., one that can encode arithmetic).[2]

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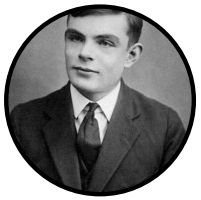

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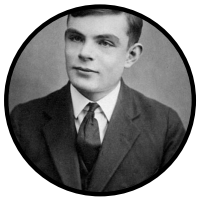

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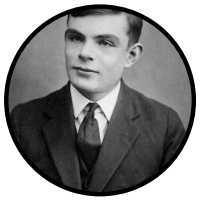

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- **Turing Machine** is the origin of **computers**.
- **Lambda Calculus** is the origin of **programming languages**.

Universal Turing Machine (UTM)

PLRG

- **Alan Turing**'s definition of computation – **Turing Machines (TMs)**.

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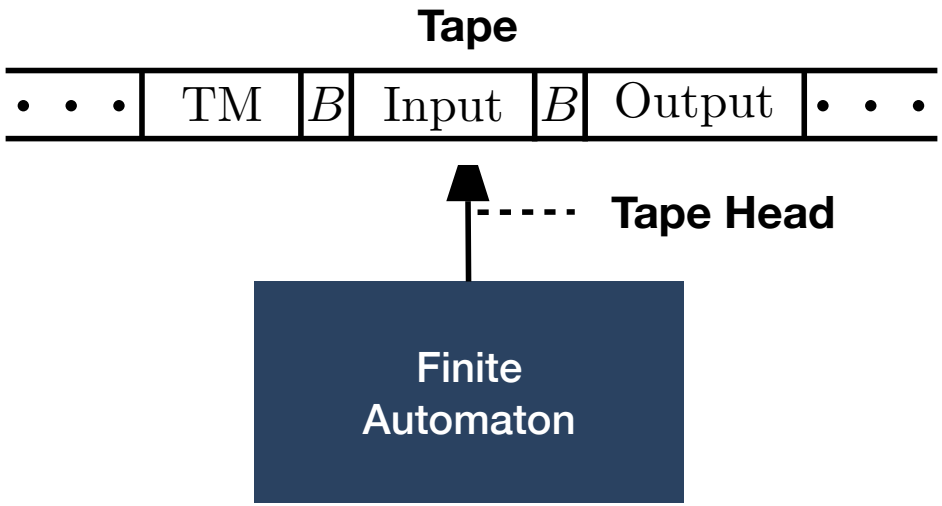

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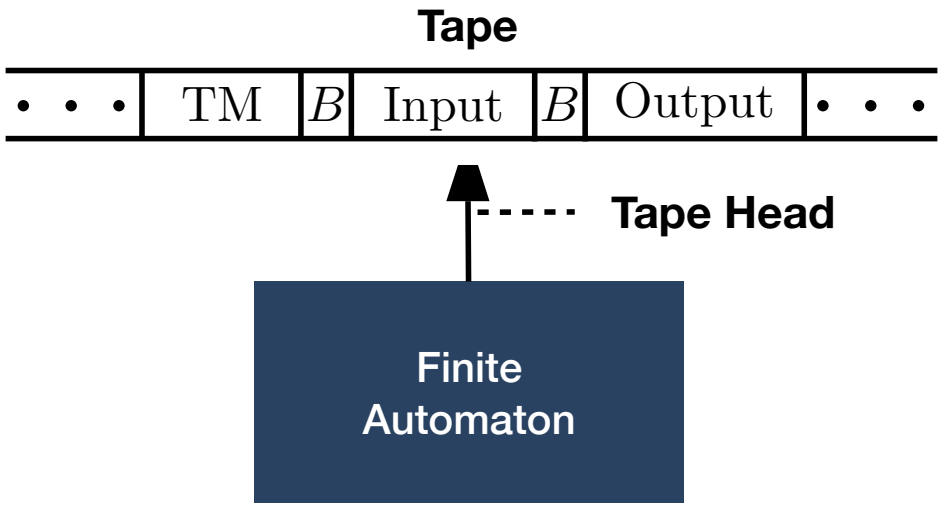

- **UTM** was **the most important invention in computer science** because it was the first time we can write a **program (software)** instead of building a new **machine (hardware)** to solve a new problem.

Disproof using Turing Machine

- Assume a TM A solves the **Decision Problem**.

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- We can build a TM H that solves the **Halting Problem** by using A:

$$\forall \text{ TM } M.\ \forall w \in a^*.\ H(M, w) = \begin{cases} \texttt{halt} & \text{if } A(\text{“}M \text{ halts on } w\text{”}) \\ \texttt{loop} & \text{otherwise} \end{cases}$$

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- Consider the following enumeration of TMs:

$H(M_i, w_i)$	w_1	w_2	w_3	$\cdots$
M_1	`halt`	`loop`	`halt`	$\cdots$
M_2	`halt`	`halt`	`loop`	$\cdots$
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- Then, F is not in the enumeration (i.e., $F \neq M_i$ for all i). It contradicts the **enumerability of TMs**. So, A **does not exist.**

Lambda Calculus

- **Alonzo Church**'s definition of computation is the **Lambda Calculus (LC)**:

$$
\begin{array}{lcll}
\Lambda \ni E & ::= & x & \text{(Variable)} \\
& | & \lambda x.\ E & \text{(Abstraction)} \\
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$$(\lambda x.\ E)\ E' \rightarrow E[x \mapsto E']$$

- For example,

$$(\lambda x.\ (\lambda y.\ x\ y))\ z \rightarrow \lambda y.\ z\ y$$

Lambda Calculus

PLRG

- **Alonzo Church**'s definition of computation is the **Lambda Calculus (LC)**:

$$\begin{array}{rcll} \Lambda \ni E & ::= & x & \text{(Variable)} \\ & | & \lambda x.\ E & \text{(Abstraction)} \\ & | & E\ E & \text{(Application)} \end{array}$$

- **Computations** are done by β-**reduction**:

$$(\lambda x.\ E)\ E' \rightarrow E[x \mapsto E']$$

- For example,

$$(\lambda x.\ (\lambda y.\ x\ y))\ z \rightarrow \lambda y.\ z\ y$$

- Every **computable function** can be represented by a **lambda term**.
- If there is no more possible β-reduction, the term is in **normal form**.

Lambda Calculus – Church Encoding

- However, there is no **data structures** or **control flows** in LC.

Lambda Calculus – Church Encoding

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- Surprisingly, we can **encode** them – **Church Encoding**:

Lambda Calculus – Church Encoding

PLRG

- However, there is no **data structures** or **control flows** in LC.
- Surprisingly, we can **encode** them – **Church Encoding**:

Boolean Values and Operations

$$\begin{aligned}
\text{true} &= \lambda x.\ \lambda y.\ x \\
\text{false} &= \lambda x.\ \lambda y.\ y \\
\text{and} &= \lambda b_1.\ \lambda b_2.\ b_1\ b_2\ \text{false} \\
\text{or} &= \lambda b_1.\ \lambda b_2.\ b_1\ \text{true}\ b_2
\end{aligned}$$

Natural Numbers and Operations

$$\begin{aligned}
0 &= \lambda f.\ \lambda x.\ x \\
1 &= \lambda f.\ \lambda x.\ f\ x \\
2 &= \lambda f.\ \lambda x.\ f\ (f\ x) \\
3 &= \lambda f.\ \lambda x.\ f\ (f\ (f\ x)) \\
\text{plus} &= \lambda n_1.\ \lambda n_2.\ \lambda f.\ \lambda x.\ n_1\ f\ (n_2\ f\ x) \\
\text{times} &= \lambda n_1.\ \lambda n_2.\ \lambda f.\ \lambda x.\ n_1\ (n_2\ f)\ x \\
\text{exp} &= \lambda n_1.\ \lambda n_2.\ n_2\ n_1
\end{aligned}$$

Control Flows

$$\begin{aligned}
\text{if} &= \lambda b.\ \lambda e_1.\ \lambda e_2.\ b\ e_1\ e_2 \\
\text{Y} &= \lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))
\end{aligned}$$

Pairs

$$\begin{aligned}
\text{pair} &= \lambda x.\ \lambda y.\ \lambda f.\ f\ x\ y \\
\text{fst} &= \lambda p.\ p\ (\lambda x.\ \lambda y.\ x) \\
\text{snd} &= \lambda p.\ p\ (\lambda x.\ \lambda y.\ y)
\end{aligned}$$

Lists

$$\begin{aligned}
\text{nil} &= \lambda c.\ \lambda n.\ n \\
\text{cons} &= \lambda h.\ \lambda t.\ \lambda c.\ \lambda n.\ c\ h\ (t\ c\ n) \\
\text{head} &= \lambda l.\ l\ (\lambda h.\ \lambda t.\ h) \\
\text{isnil} &= \lambda l.\ l\ (\lambda h.\ \lambda t.\ \text{false})\ \text{true}
\end{aligned}$$

Lambda Calculus – Church Encoding

PLRG

$$0 = \lambda f.\ \lambda x.\ x \qquad \text{plus} = \lambda n_1.\ \lambda n_2.\ \lambda f.\ \lambda x.\ n_1\ f\ (n_2\ f\ x)$$
$$1 = \lambda f.\ \lambda x.\ f\ x \qquad \text{times} = \lambda n_1.\ \lambda n_2.\ \lambda f.\ \lambda x.\ n_1\ (n_2\ f)\ x$$
$$2 = \lambda f.\ \lambda x.\ f\ (f\ x) \qquad \text{exp} = \lambda n_1.\ \lambda n_2.\ n_2\ n_1$$
$$3 = \lambda f.\ \lambda x.\ f\ (f\ (f\ x))$$

For example, we can compute $1 + 1$ as follows:

$$\begin{aligned}
\text{plus}\ 1\ 1 &= (\lambda n_1.\ \lambda n_2.\ \lambda f.\ \lambda x.\ n_1\ f\ (n_2\ f\ x))\ 1\ 1 \\
&\to \lambda f.\ \lambda x.\ 1\ f\ (1\ f\ x) \\
&= \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ ((\lambda f.\ \lambda x.\ f\ x)\ f\ x) \\
&\to \lambda f.\ \lambda x.\ (\lambda f.\ \lambda x.\ f\ x)\ f\ (f\ x) \\
&\to \lambda f.\ \lambda x.\ f\ (f\ x) \\
&= 2
\end{aligned}$$

The **normal form** (computational result) of (plus 1 1) is 2.

Disproof using Lambda Calculus

PLRG

- Church proved that there is **no computable function** that can decide whether two **lambda terms** are **equivalent** or not:

$$\nexists\, \text{eq?} \in \Lambda.\ \forall\, E_1, E_2 \in \Lambda.\ (\text{eq?}\ E_1\ E_2) \rightarrow \begin{cases} \text{true} & \text{if } E_1 \equiv E_2 \\ \text{false} & \text{otherwise} \end{cases}$$

where $E_1 \equiv E_2$ means E_1 and E_2 are equivalent, i.e., they have the same **normal form** (computational result).

Disproof using Lambda Calculus

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- For example, (plus 1 1) and (plus 0 2) are equivalent in LC because they have the same normal form 2.

Disproof using Lambda Calculus

PLRG

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  where $E_1 \equiv E_2$ means E_1 and E_2 are equivalent, i.e., they have the same **normal form** (computational result).
- For example, (plus 1 1) and (plus 0 2) are equivalent in LC because they have the same normal form 2.
- It means that there is no computable function that can **decide** whether a **lambda term** has a given **normal form** or not.
- We skip the proof here.

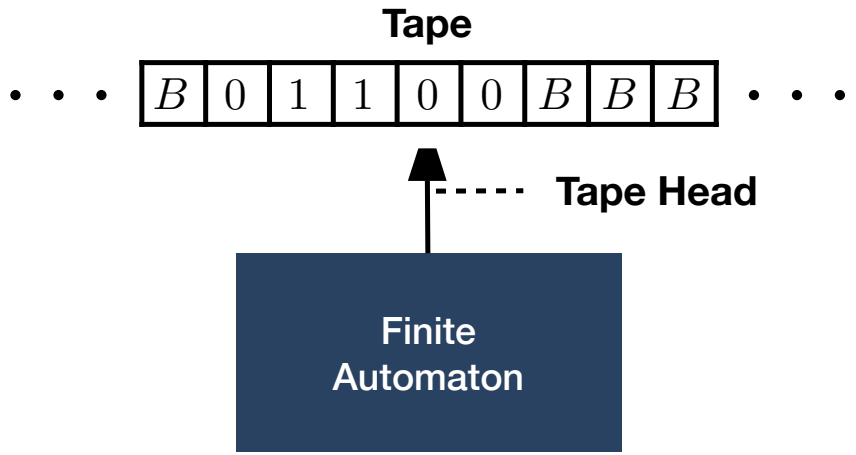
Contents

1. Gödel's Incompleteness Theorem
    - Example: Continuum Hypothesis
    - Gödel Numbering

2. Entscheidungsproblem – Decision Problem
    - Disproof using Turing Machine
    - Disproof using Lambda Calculus

3. Church-Turing Thesis

Church-Turing Thesis



(Turing Machine)

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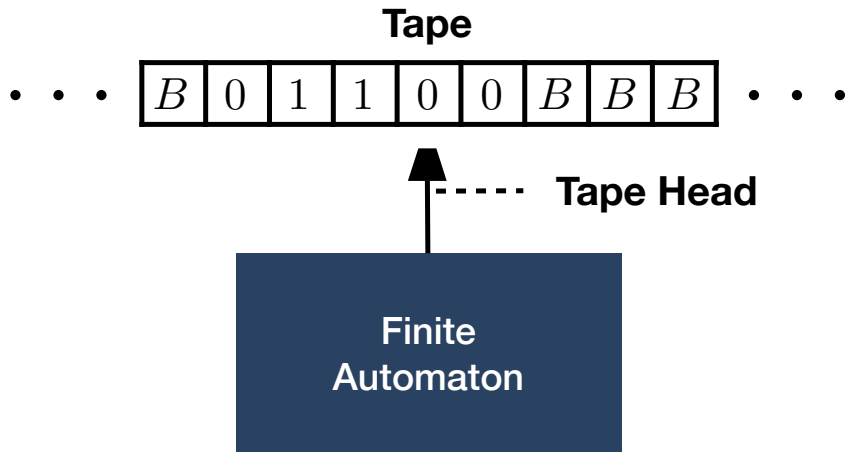
(Lambda Calculus)

- **LC** has the same computational power as **TMs**. (**Turing Complete**)

Church-Turing Thesis



(Turing Machine)

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(Lambda Calculus)

- **LC** has the same computational power as **TMs**. (**Turing Complete**)
- **Church-Turing Thesis**:

  Any **real-world computation** *can be translated into an equivalent computation involving a* **Turing machine** *or can be done using* **lambda calculus**.

Summary

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   - Example: Continuum Hypothesis
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   - Disproof using Turing Machine
   - Disproof using Lambda Calculus

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Next Lecture

- Undecidability

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