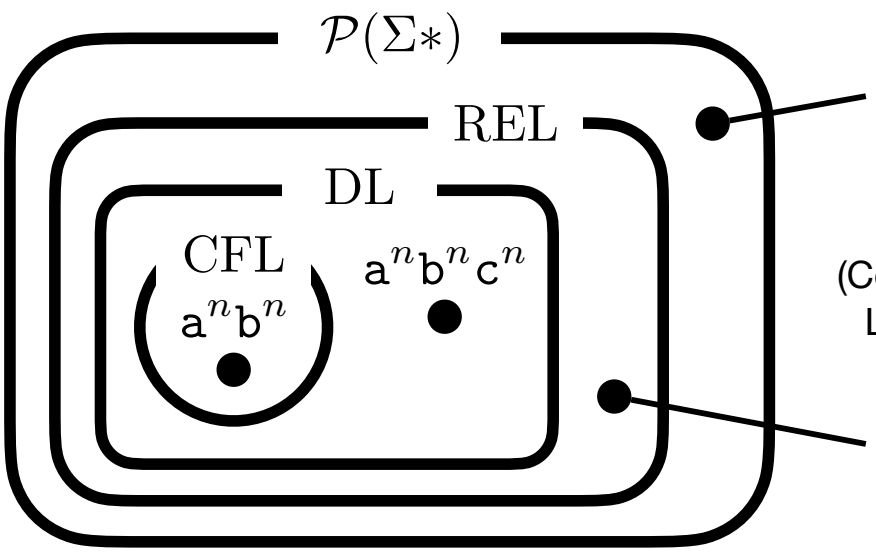
Lecture 26 – P, NP, and NP-Complete Problems

COSE215: Theory of Computation

Jihyeok Park

2026 Spring

Recall



$$\underset{\text{(Context-Free Language)}}{\text{CFL}} \subset \underset{\text{(Decidable Language)}}{\text{DL}} \subset \underset{\text{(Recursively Enumerable Language)}}{\text{REL}}$$

Definition (Decision Problem)

A **decision problem** π is a computational problem whose answer is either **yes** or **no** for a given input.

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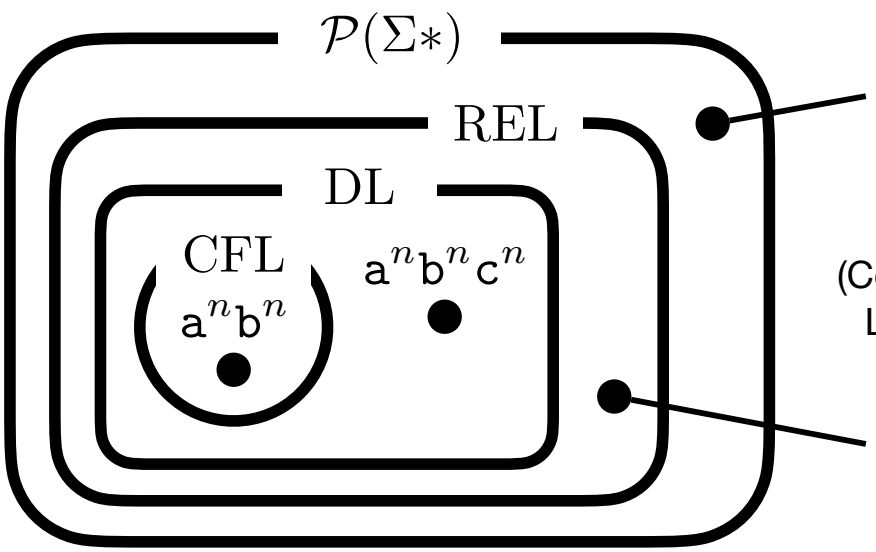

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In this lecture, we will **classify** decision problems based on the **time complexity** of possible TMs (or NTMs) that solve the problems.

Contents

PLRG

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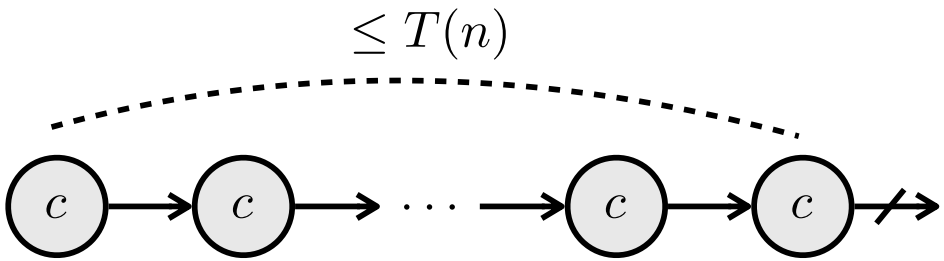
Time Complexity of TMs

Definition (Time Complexity of TMs)

We say a **Turing machine (TM)** M has a **time complexity** $T : \mathbb{N} \to \mathbb{N}$ if M halts on w in at most $T(n)$ moves for all $w \in \Sigma^*$ whose length is n.



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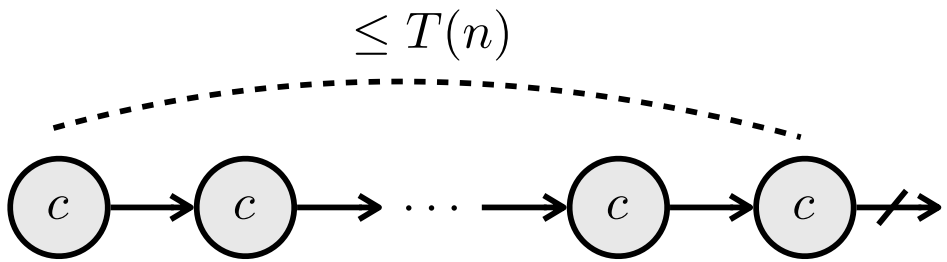

Definition (**DTIME**)

A decision problem π is in **DTIME**$(T(n))$ if it is decidable by a TM M whose time complexity is $T(n)$.

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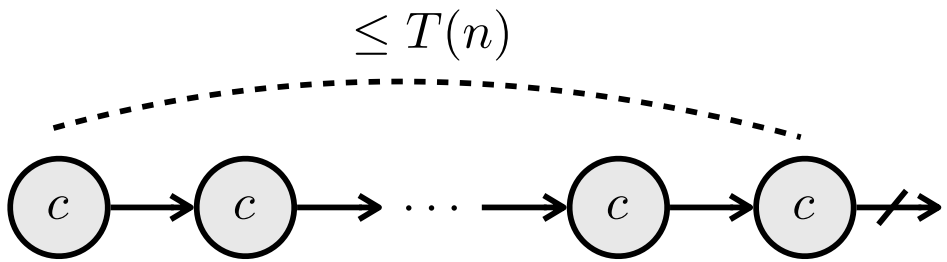

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We often use a **big** O **notation** to describe the time complexity of a TM:

$$f(n) = O(g(n)) \iff \exists k \in \mathbb{N}, n_0 \in \mathbb{N}.\ \forall n \geq n_0.\ f(n) \leq k \cdot g(n)$$

Time Complexity of TMs – Example

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$\langle$**EvenPalin**$\rangle$ – Is a word $w \in \{\mathtt{a}, \mathtt{b}\}^*$ an even-length palindrome?

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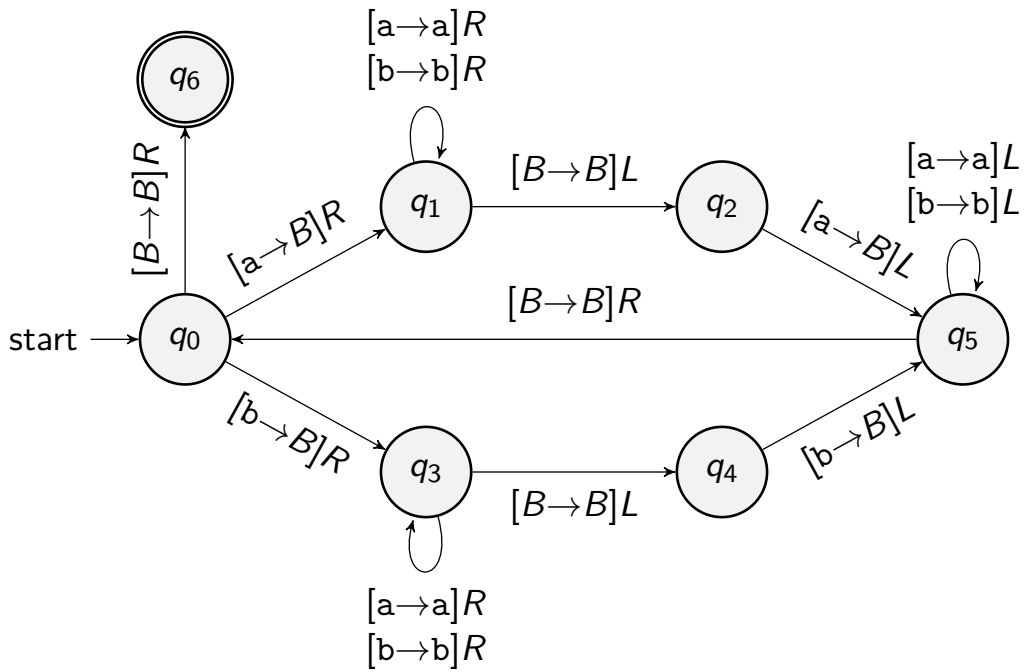

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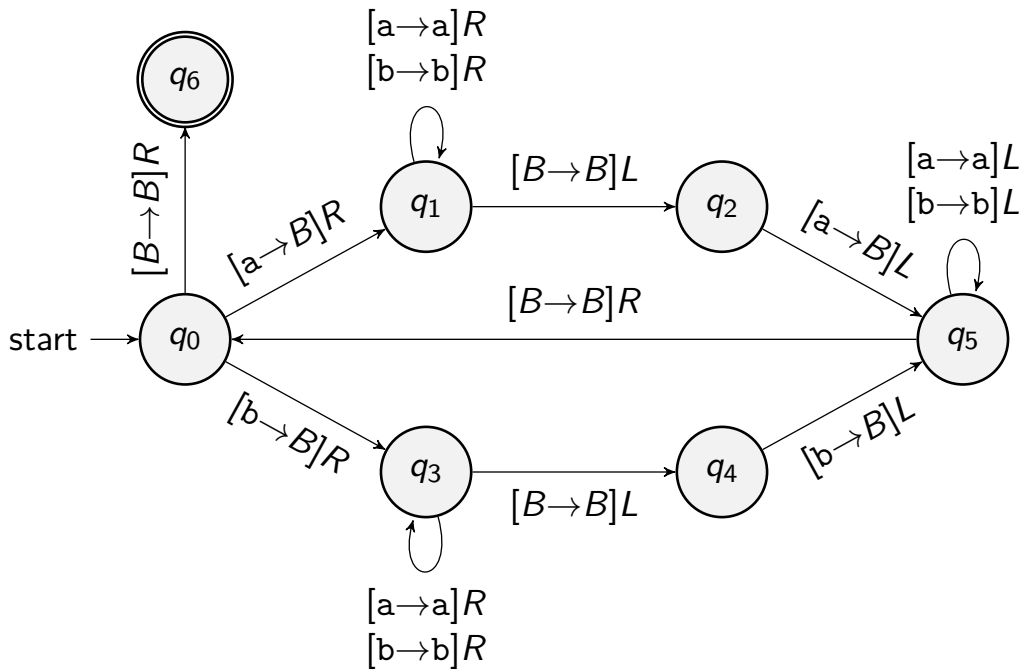

The decision problem $\langle$**EvenPalin**$\rangle$ is decidable by the above TM whose time complexity is $T(n) = (n+1)(n+2)/2 = O(n^2)$.

$$\langle\textbf{EvenPalin}\rangle \in \textbf{DTIME}(O(n^2))$$

P – Polynomial Time Complexity

PLRG

Definition (**P** – Polynomial Time Complexity)

A decision problem π is in **P** if it is decidable by a TM M whose time complexity is a **polynomial function** (i.e., $T(n) = O(n^k)$ for some $k \geq 0$).

$$\mathbf{P} = \bigcup_{k \geq 0} \mathbf{DTIME}(O(n^k))$$

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For example, the decision problem $\langle \mathbf{EvenPalin} \rangle$ is in **P**.

$$\langle \mathbf{EvenPalin} \rangle \in \mathbf{DTIME}(O(n^2)) \subseteq \mathbf{P}$$

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Definition (Tractable Problems)

A problem π is called a **tractable problem** if it is a **P** problem.

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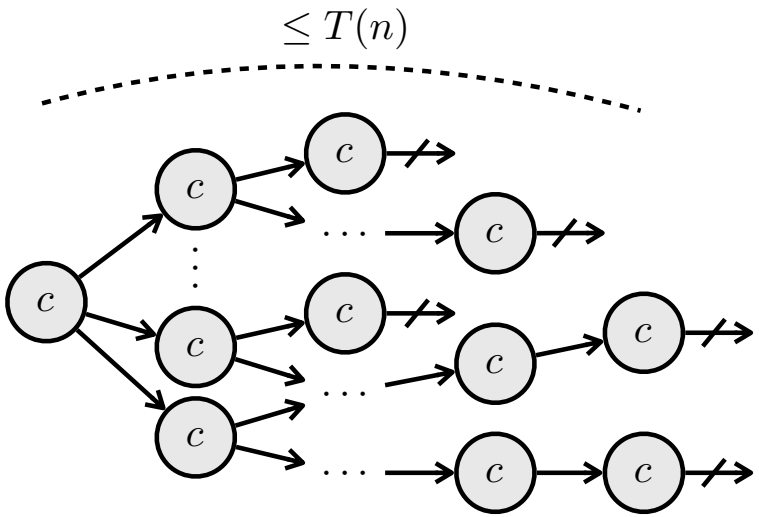
We say a **nondeterministic Turing machine (NTM)** M has a **time complexity** $T : \mathbb{N} \to \mathbb{N}$ if M halts on w in at most $T(n)$ moves for all $w \in \Sigma^*$ whose length is n.



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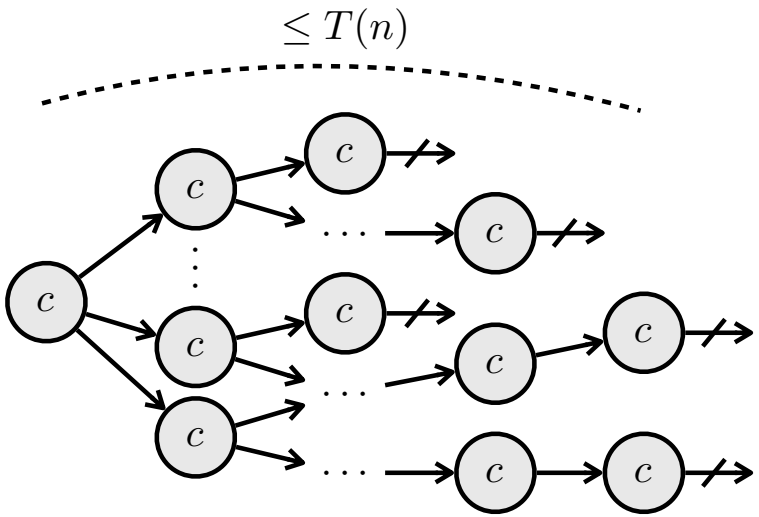

Definition (**NTIME**)

A decision problem π is in **NTIME**$(T(n))$ if it is decidable by a NTM M whose time complexity is $T(n)$.

Time Complexity of NTMs – Example

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$\langle$**MakeEvenPalin**$\rangle$ – Is a word $w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^*$ convertible to an even-length palindrome by replacing all $\mathtt{c}$'s with $\mathtt{a}$'s or $\mathtt{b}$'s?

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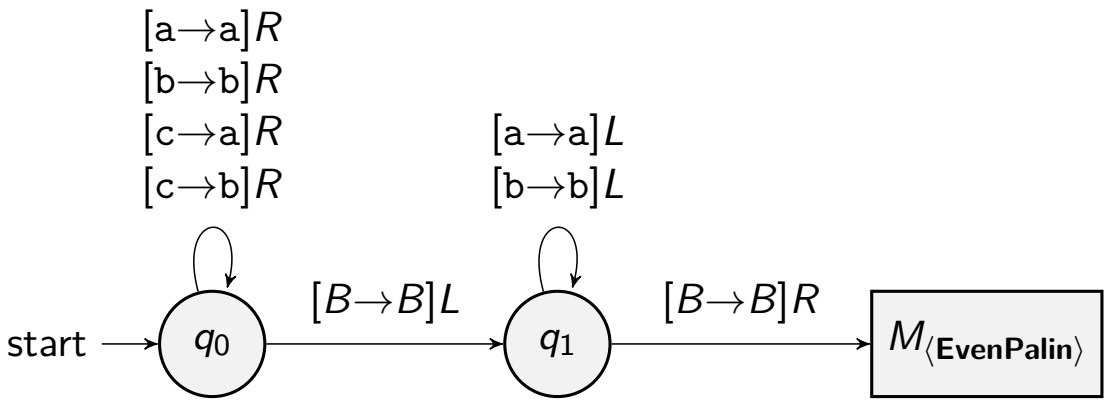

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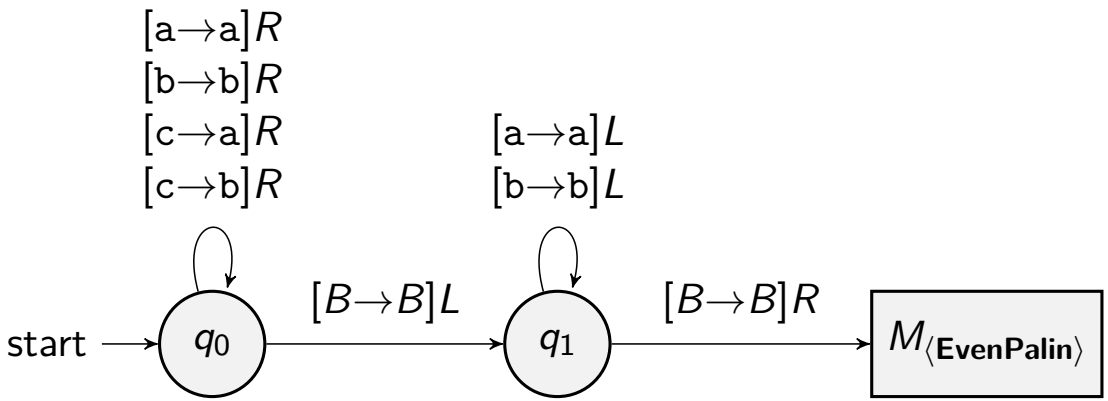

The decision problem $\langle$**MakeEvenPalin**$\rangle$ is decidable by the above NTM whose time complexity is $T(n) = 2(n+1) + O(n^2) = O(n^2)$.

$$\langle\textbf{MakeEvenPalin}\rangle \in \textbf{NTIME}(O(n^2))$$

NP – Nondeterministic Polynomial Time Complexity

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Definition (**NP** – Nondeterministic Polynomial Time Complexity)

A decision problem π is in **NP** if it is decidable by an NTM M whose time complexity is a **polynomial function** (i.e., $T(n) = O(n^k)$ for some $k \geq 0$).

$$\mathbf{NP} = \bigcup_{k \geq 0} \mathbf{NTIME}(O(n^k))$$

For example, the decision problem $\langle \mathbf{MakeEvenPalin} \rangle$ is in **NP**.

$$\langle \mathbf{MakeEvenPalin} \rangle \in \mathbf{NTIME}(O(n^2)) \subseteq \mathbf{NP}$$

Search Problem

PLRG

Definition (Search Problem)

A **search problem** π is a decision problem that asks for the existence of a **witness** x (i.e., a solution) in the search space $S(w)$ for a given input w, satisfying another decision problem π' as a **verification problem**.

$$\forall w \in \Sigma^*.\ \pi(w) = \text{yes} \iff \exists x \in S(w).\ \pi'(w, x) = \text{yes}$$

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where the search space $S(w)$ of an input w is defined as follows:

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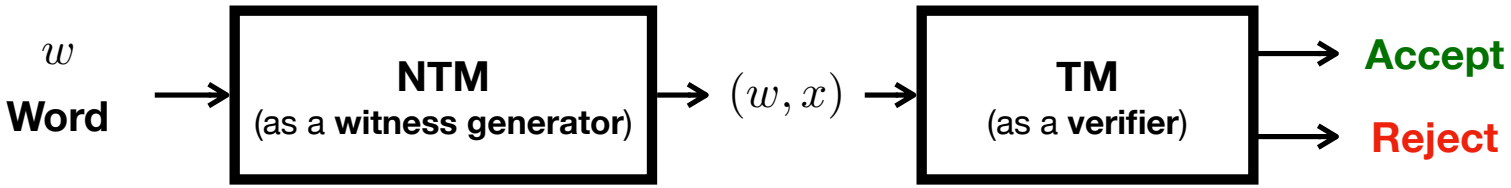
$$\text{e.g.,}\quad w = \texttt{acbcca}\quad S(w) = \left\{ \begin{array}{llll} \texttt{aabaaa}, & \texttt{aababa}, & \texttt{aabbaa}, & \texttt{aabbba}, \\ \texttt{abbaaa}, & \texttt{abbaba}, & \texttt{abbbaa}, & \texttt{abbbba} \end{array} \right\}$$

Definition (**NP** – Verifier-based Definition)

A search problem π defined with a verification problem π' is in **NP** if there is a polynomial time TM M as a **verifier** for π:

$$\forall w \in \Sigma^*.\ \forall x \in S(w).\ \pi'(w, x) = \text{yes} \iff (w, x) \in L(M)$$



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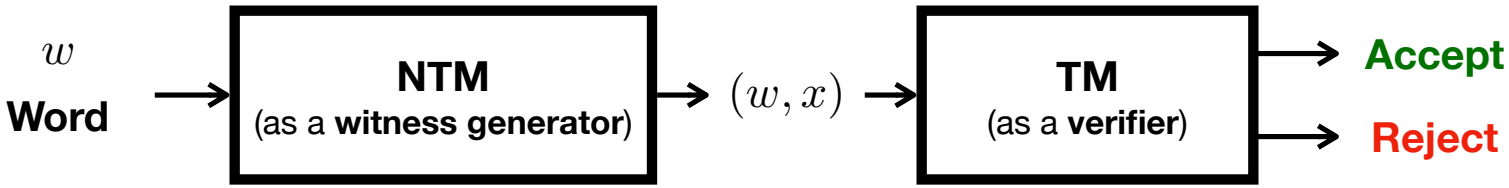

For example, $\langle$**MakeEvenPalin**$\rangle$ is a search problem in **NP**:

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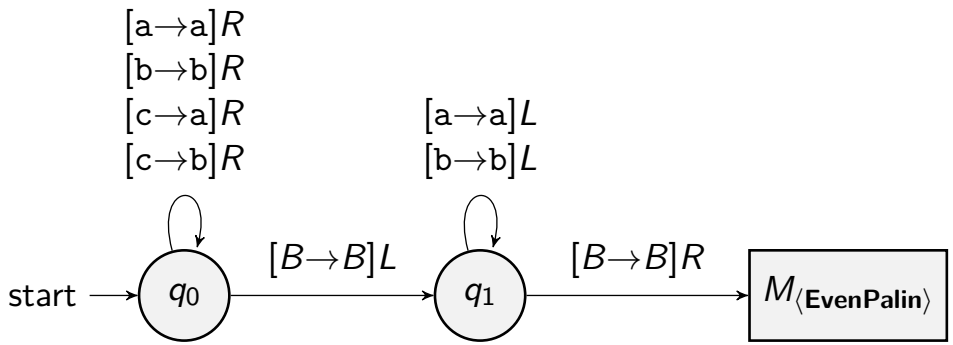

NP – Example: $\langle$**SAT**$\rangle$

$\langle$**SAT**$\rangle$ (Boolean **SAT**isfiability problem) – Is a given Boolean formula (consisting of Boolean variables, $\wedge$, $\vee$, and $\neg$) satisfiable?

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For example, is the following Boolean formula satisfiable?

$$(x_1 \vee \neg x_2) \wedge (x_1 \vee x_2 \vee x_3) \wedge \neg x_1$$

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Yes! For example, $x_1 = \texttt{\#f}$, $x_2 = \texttt{\#f}$, and $x_3 = \texttt{\#t}$ is a satisfying assignment.

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We can construct a polynomial time TM as a **verifier** for ⟨**SAT**⟩, which takes 1) a **Boolean formula** and 2) an **assignment** of Boolean variables, and checks whether the assignment satisfies the formula.

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We can construct a polynomial time TM as a **verifier** for ⟨**SAT**⟩, which takes 1) a **Boolean formula** and 2) an **assignment** of Boolean variables, and checks whether the assignment satisfies the formula.

In other words, we can construct a polynomial time NTM for ⟨**SAT**⟩ by 1) **generating all assignments** of Boolean variables and 2) **verifying** whether the assignment satisfies the formula using the verifier.

Contents

Polynomial Time Reduction ($\leq_P$)

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Definition (Polynomial Time Reduction ($\leq_P$))

A decision problem π_1 is **polynomial time reducible** to another decision problem π_2 (denoted by $\pi_1 \leq_P \pi_2$) if there exists a polynomial time computable function $f : \Sigma^* \to \Sigma^*$ such that:

$$\forall w \in \Sigma^*.\ \pi_1(w) = \text{yes} \iff \pi_2(f(w)) = \text{yes}$$

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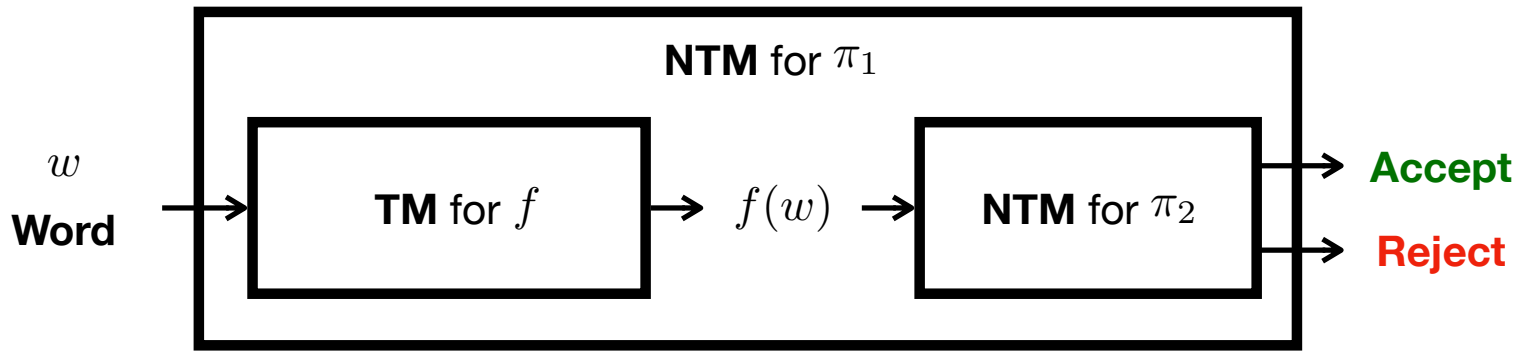
We say that π_2 is **harder** than π_1 if $\pi_1 \leq_P \pi_2$ because we can solve π_1 in polynomial time if we can solve π_2 in polynomial time.



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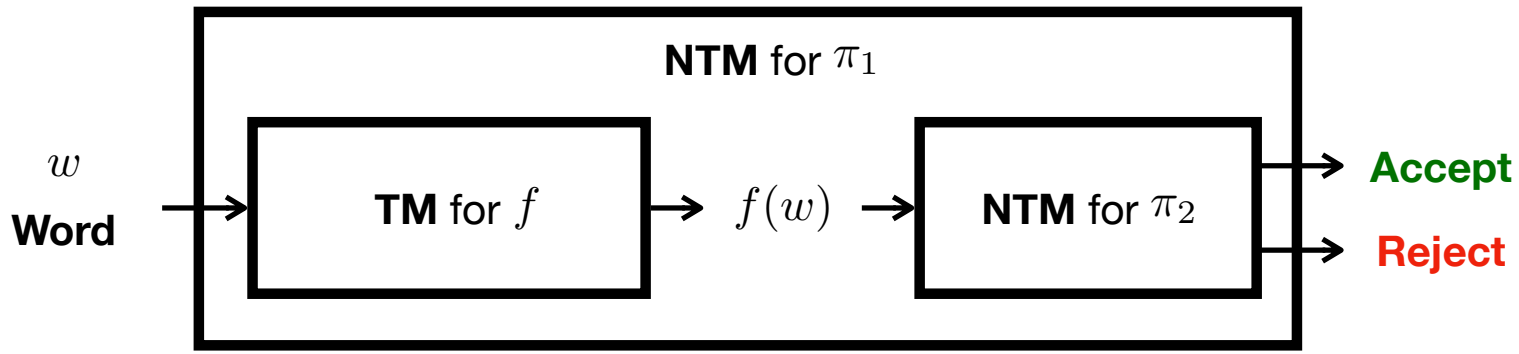

If a decision problem π_2 is in **NP** and $\pi_1 \leq_P \pi_2$, then π_1 is in **NP**.

Polynomial Time Reduction ($\leq_P$) – Example

PLRG

Consider the following two decision problems:

- $\langle$**MakeEvenPalin**$\rangle$ – Is a word $w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^*$ convertible to an even-length palindrome by replacing all $\mathtt{c}$'s with $\mathtt{a}$'s or $\mathtt{b}$'s?
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We can show that $\langle$**MakeEvenPalin**$\rangle \leq_P \langle$**SAT**$\rangle$ by the following polynomial time computable function f:

$$f(a_1 a_2 \cdots a_n) = \bigwedge_{i=1}^{n}((x_i \wedge x_{n+1-i}) \vee (\neg x_i \wedge \neg x_{n+1-i})) \\ \wedge \bigwedge\{x_i \mid a_i = \mathtt{a}\} \wedge \bigwedge\{\neg x_i \mid a_i = \mathtt{b}\}$$

($x_i = \#\mathtt{t}$ means that $a_i = \mathtt{a}$ and $x_i = \#\mathtt{f}$ means that $a_i = \mathtt{b}$)

Polynomial Time Reduction ($\leq_P$) – Example

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For example,

$$f(\mathtt{acba}) = ((x_1 \wedge x_4) \vee (\neg x_1 \wedge \neg x_4)) \wedge ((x_2 \wedge x_3) \vee (\neg x_2 \wedge \neg x_3)) \wedge x_1 \wedge \neg x_3 \wedge x_4$$

Polynomial Time Reduction ($\leq_P$) – Example

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Consider the following two decision problems:

- $\langle$**MakeEvenPalin**$\rangle$ – Is a word $w \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}^*$ convertible to an even-length palindrome by replacing all c's with a's or b's?
- $\langle$**SAT**$\rangle$ – Is a given Boolean formula satisfiable?

We can show that $\langle$**MakeEvenPalin**$\rangle \leq_P \langle$**SAT**$\rangle$ by the following polynomial time computable function f:

$$f(a_1 a_2 \cdots a_n) = \bigwedge_{i=1}^{n}((x_i \wedge x_{n+1-i}) \vee (\neg x_i \wedge \neg x_{n+1-i})) \\ \wedge \bigwedge\{x_i \mid a_i = \mathtt{a}\} \wedge \bigwedge\{\neg x_i \mid a_i = \mathtt{b}\}$$

($x_i = \#\mathtt{t}$ means that $a_i = \mathtt{a}$ and $x_i = \#\mathtt{f}$ means that $a_i = \mathtt{b}$)

For example,

$$f(\mathtt{acba}) = ((x_1 \wedge x_4) \vee (\neg x_1 \wedge \neg x_4)) \wedge ((x_2 \wedge x_3) \vee (\neg x_2 \wedge \neg x_3)) \wedge x_1 \wedge \neg x_3 \wedge x_4$$

Thus, we can solve $\langle$**MakeEvenPalin**$\rangle$ using a machine for $\langle$**SAT**$\rangle$, and $\langle$**SAT**$\rangle$ is harder problem than $\langle$**MakeEvenPalin**$\rangle$.

NP-complete – Hardest Problems in **NP**

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Definition (**NP-hard** – Harder Problems Than All **NP**)

A decision problem π is in **NP-hard** if $\forall \pi' \in \mathbf{NP}$, $\pi' \leq_P \pi$.

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⟨**SAT**⟩ – The First **NP-complete** Problem

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Theorem (Cook–Levin theorem)

⟨***SAT***⟩ *is in* ***NP-complete****.*

[1] https://en.wikipedia.org/wiki/Cook-Levin_theorem

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For ②, we need to show that $\forall \pi \in$ **NP**, $\pi \leq_P \langle$**SAT**$\rangle$.

The core idea is to simulate an NTM M for π using a Boolean formula ϕ such that ϕ is satisfiable if and only if M accepts w. But, we skip the details of the proof. Please refer to the link[1] for the details.

[1] https://en.wikipedia.org/wiki/Cook-Levin_theorem

Other **NP-complete** Problems

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Theorem (Lemma)

*A decision problem π is in **NP-hard** if $\langle \textbf{SAT} \rangle \leq_P \pi$*

[2] https://en.wikipedia.org/wiki/List_of_NP-complete_problems

Other **NP-complete** Problems

PLRG

Theorem (Lemma)

*A decision problem π is in **NP-hard** if $\langle \textbf{SAT} \rangle \leq_P \pi$*

This lemma is very useful to show that a decision problem π is in **NP-complete** by showing that 1) π is in **NP** and 2) $\langle \textbf{SAT} \rangle \leq_P \pi$.

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PLRG

Theorem (Lemma)

*A decision problem π is in **NP-hard** if $\langle \textbf{SAT} \rangle \leq_P \pi$*

This lemma is very useful to show that a decision problem π is in **NP-complete** by showing that 1) π is in **NP** and 2) $\langle \textbf{SAT} \rangle \leq_P \pi$.

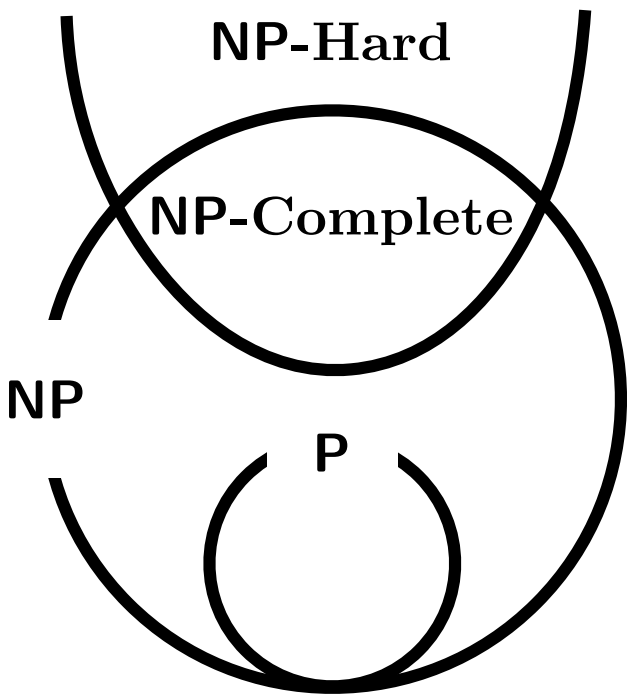
We can show that all of the following decision problems are in **NP-complete** by using this lemma:[2]

- $\langle \textbf{SubsetSum} \rangle$ – Given a set of integers S and an integer t, is there a subset $S' \subseteq S$ such that $\sum S' = t$?
- $\langle \textbf{Clique} \rangle$ – Given a graph G and an integer k, is there a clique of size k in G?
- $\langle \textbf{VertexCover} \rangle$ – Given a graph G and an integer k, is there a vertex cover of size k in G?
- ...

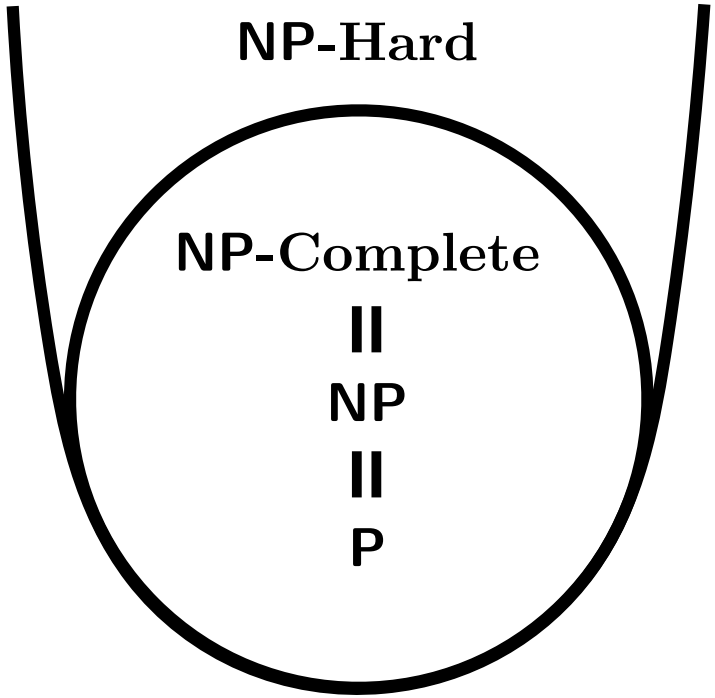
[2] https://en.wikipedia.org/wiki/List_of_NP-complete_problems

Contents

Major Unsolved Problem: P = NP?



P ≠ NP



P = NP

"If P = NP, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in creative leaps, no fundamental gap between solving a problem and recognizing the solution once it's found."

— Scott Aaronson, *UT Austin*

Summary

PLRG

1. **P**
   - Time Complexity of TMs
   - **P** – Polynomial Time Complexity (Tractable Problems)
2. **NP**
   - Time Complexity of NTMs
   - **NP** – Nondeterministic Polynomial Time Complexity
   - **NP** – Verifier-based Definition
3. **NP-complete**
   - Polynomial Time Reduction ($\leq_P$)
   - **NP-complete** – Hardest Problems in **NP**
   - $\langle$**SAT**$\rangle$ – The First **NP-complete** Problem
   - Other **NP-complete** Problems
4. Major Unsolved Problem: **P** = **NP**?

Final Exam

PLRG

- The final exam will be given in class.
- **Date:** 13:30-14:45 (1 hour 15 minutes), June 17 (Wed.).
- **Location:**
  - **IT-Education Hall B102:** Students with Student ID 2025XXXX
  - **IT-Education Hall 611:** All other students
- **Coverage:** Lectures 14 – 26
- **Format:** 7–9 questions with closed book and closed notes
  - Yes/No questions about concepts in the theory of computation.
  - Construction of automata or grammars for given languages.
  - Proofs of given statements related to automata or grammars.
  - etc.
- Note that there is **no class** on **June 22 (Mon.)**.
- Please refer to the **previous exams** in the course website:

https://plrg.korea.ac.kr/courses/cose215/

Next Lecture

- Course Review

Jihyeok Park
jihyeok_park@korea.ac.kr
https://plrg.korea.ac.kr