Lecture 9 – The Pumping Lemma for Regular Languages

COSE215: Theory of Computation

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PLRG

2026 Spring

Recall

The class of regular languages is closed under the following operations:

- Union
- Concatenation
- Kleene Star
- Complement
- Intersection
- Difference
- Reversal
- Homomorphism
- Inverse Homomorphism

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For example, consider $L_i = \{\mathtt{a}^i\mathtt{b}^i\}$ for $i \geq 0$. Then, L_i and its complement L_i^c are regular languages. But, their infinite union and intersection are not regular languages:

$$\bigcup_{i \geq 0} L_i = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\} = L \qquad \bigcap_{i \geq 0} L_i^c = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}^c = L^c$$

because $L = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$ is not regular.

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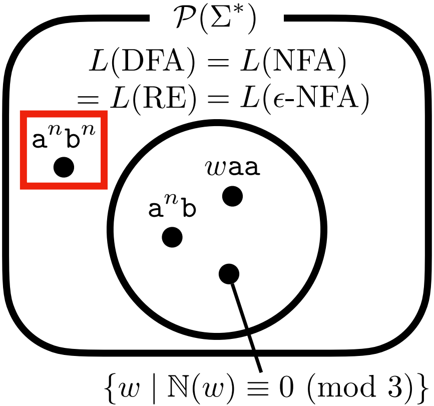
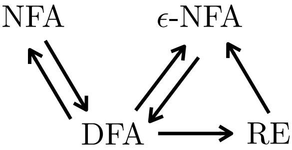
- Not all languages are regular: e.g., $L = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$.





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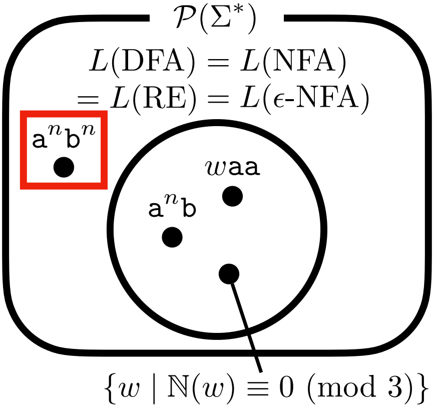

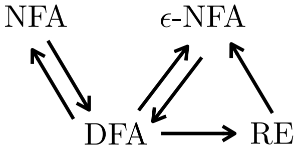

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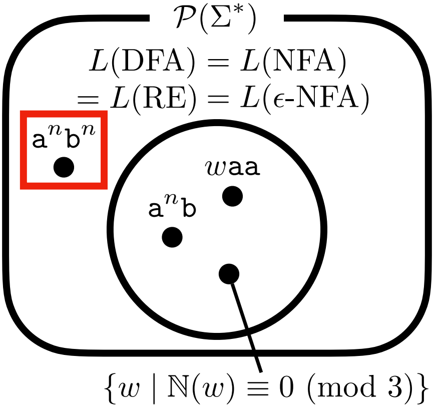

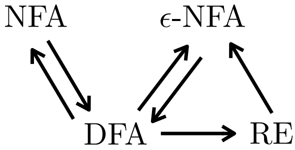

- How to prove that a language is **NOT** regular? **Pumping Lemma!**

Contents

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   The **Pumping Lemma** formally captures this intuition.

Pumping Lemma for Regular Languages

Lemma (Pumping Lemma for Regular Languages)

For a given regular language L, **there exists** *a positive integer n such that* **for all** $w \in L$, *if* $|w| \geq n$, **there exists** $w = xyz$ *such that*

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$$q_0 = p_0 \xrightarrow{\mathtt{a}_1} p_1 \xrightarrow{\mathtt{a}_2} \cdots \xrightarrow{\mathtt{a}_n} p_n \xrightarrow{\mathtt{a}_{n+1}} \cdots \xrightarrow{\mathtt{a}_m} p_m \in F$$

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$$① - |y| = j - i > 0 \qquad |xy| = j \leq n - ②$$

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$$\begin{aligned} A &= \textit{L is regular} \\ & \Downarrow \\ B &= \exists n > 0.\ \forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③ \end{aligned}$$

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$$\begin{aligned} \neg B &= \neg(\exists n > 0.\ \forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③) \\ &= \forall n > 0.\ \neg(\forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③) \\ &= \forall n > 0.\ \exists w \in L.\ \neg(|w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③) \\ &= \forall n > 0.\ \exists w \in L.\ |w| \geq n \land \neg(\exists w = xyz.\ ① \land ② \land ③) \\ &= \forall n > 0.\ \exists w \in L.\ |w| \geq n \land \forall w = xyz.\ \neg(① \land ② \land ③) \end{aligned}$$

Lemma (Pumping Lemma for Regular Languages)

$$A = L \text{ is regular}$$
$$\Downarrow$$
$$B = \exists n > 0.\ \forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \wedge ② \wedge ③$$

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Lemma (Pumping Lemma for Regular Languages)

$$\begin{array}{rcl} A & = & L \textit{ is regular} \\ & & \Downarrow \\ B & = & \exists n > 0.\ \forall w \in L.\ |w| \geq n \Rightarrow \exists w = xyz.\ ① \land ② \land ③ \end{array}$$

$$\begin{array}{rcll} A & \Longrightarrow & B & (O) \\ B & \Longrightarrow & A & (X) \\ \neg B & \Longrightarrow & \neg A & (O) \end{array}$$

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Application: Proving Languages are Not Regular

To prove a language L is **NOT** regular, we need to show that

$$\forall n > 0.\ \exists w \in L.\ |w| \geq n \land \forall w = xyz.\ (①\land ②) \Rightarrow \neg ③$$

1. $|y| > 0$
2. $|xy| \leq n$
3. $\forall i \geq 0.\ xy^iz \in L$

Note that $\neg③ = \ \exists i \geq 0.\ xy^iz \notin L$.

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Note that $\neg ③ = \exists i \geq 0.\ xy^iz \notin L$.

We can prove this by following the steps below:

1. Assume any positive integer n is given.
2. Pick a word $w \in L$.
3. Show that $|w| \geq n$.
4. Assume any split $w = xyz$ is given, and ① $|y| > 0 \land$ ② $|xy| \leq n$.
5. $\neg$③ Pick $i \geq 0$, and show that $xy^iz \notin L$ using ① and ②.

Contents

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3. $|w| = n + n = 2n \geq n$.
4. Assume any split $w = xyz$ is given, and ① $|y| > 0 \land$ ② $|xy| \leq n$.

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Let's prove that L is **NOT** regular using the Pumping Lemma:

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2. Let $w = a^n b^n \in L$.
3. $|w| = n + n = 2n \geq n$.
4. Assume any split $w = xyz$ is given, and ① $|y| > 0 \wedge$ ② $|xy| \leq n$.
5. Let $i =$ 0 . We need to show that $\neg$③ $xy^0 z \notin L$:
   - Since ② $|xy| \leq n$,

     $$x = a^p \qquad y = a^q \qquad z = a^{n-p-q} b^n$$

     for some $0 \leq p, q \leq n$ such that $p + q \leq n$.
   - Since ① $|y| > 0$, we know $q > 0$.
   - Finally, $xy^0 z = xz = a^p a^{n-p-q} b^n = a^{n-q} b^n \notin L$ ($\because q > 0$). □

Example 2

Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{ww^R \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$$

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1. Assume any positive integer n is given.
2. Let $w = \mathtt{a}^n\mathtt{b}^n\mathtt{b}^n\mathtt{a}^n \in L$.
3. $|w| = n + n + n + n = 4n \geq n$.
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   - Since ① $|y| > 0$, we know $q > 0$.
   - Finally, $xy^0z = xz = \mathtt{a}^p\mathtt{a}^{n-p-q}\mathtt{b}^n\mathtt{b}^n\mathtt{a}^n = \mathtt{a}^{n-q}\mathtt{b}^n\mathtt{b}^n\mathtt{a}^n \notin L$ $(\because q > 0)$. □

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Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{\mathtt{a}^l \mathtt{b}^m \mathtt{c}^n \mid l + m \leq n\}$$

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Let's prove that L is **NOT** regular using the Pumping Lemma:

$$L = \{\mathtt{a}^l \mathtt{b}^m \mathtt{c}^n \mid l + m \leq n\}$$

1. Assume any positive integer n is given.
2. Let $w = \mathtt{a}^n \mathtt{b}^n \mathtt{c}^{2n} \in L$.
3. $|w| = n + n + 2n = 4n \geq n$.
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4. Assume any split $w = xyz$ is given, and ① $|y| > 0 \land$ ② $|xy| \leq n$.
5. Let $i = 2$. We need to show that $\neg$③ $xy^2z \notin L$:
   - Since ② $|xy| \leq n$,

     $$x = \mathtt{a}^p \qquad y = \mathtt{a}^q \qquad z = \mathtt{a}^{n-p-q}\mathtt{b}^n\mathtt{c}^{2n}$$

     for some $0 \leq p, q \leq n$ such that $p + q \leq n$.
   - Since ① $|y| > 0$, we know $q > 0$.
   - Finally, $xy^2z = xyyz = \mathtt{a}^{n+q}\mathtt{b}^n\mathtt{c}^{2n} \notin L$
     ($\because q > 0$. Thus, $(n + q) + n = 2n + q > 2n$). □

Example 4

Let's prove that L is **NOT** regular using the Pumping Lemma:

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Let's prove that L is **NOT** regular using the Pumping Lemma:

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2. Let $w = \mathtt{a}^{n^2} \in L$.
3. $|w| = n^2 \geq n$.
4. Assume any split $w = xyz$ is given, and ① $|y| > 0 \land$ ② $|xy| \leq n$.
5. Let $i = 2$. We need to show that $\neg$③ $xy^2z \notin L$:
   - Since ① $|y| > 0$ and ② $|xy| \leq n$,

     $$y = \mathtt{a}^k$$

     where $0 < k \leq n$. Then,

     $$n^2 < n^2 + k \ (\because 0 < k) \qquad n^2 + k < (n+1)^2 \ (\because k \leq n)$$

   - Finally, $xy^2z = xyyz = \mathtt{a}^{n^2+k} \notin L$ $\square$

Example 5

Let's prove that L is **NOT** regular:

$$L = \{\mathtt{a}^n \mathtt{b}^k \mathtt{c}^{n+k} \mid n, k \geq 0\}$$

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Let's prove that L is **NOT** regular:

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- It is much easier to use **closure properties** under **homomorphisms**.
- Consider a homomorphism $h : \{\mathtt{a}, \mathtt{b}, \mathtt{c}\} \rightarrow \{\mathtt{a}, \mathtt{b}\}^*$:

$$h(\mathtt{a}) = \mathtt{a} \qquad h(\mathtt{b}) = \mathtt{a} \qquad h(\mathtt{c}) = \mathtt{b}$$

- Then,

$$h(L) = \{\mathtt{a}^{n+k} \mathtt{b}^{n+k} \mid n, k \geq 0\} = \{\mathtt{a}^n \mathtt{b}^n \mid n \geq 0\}$$

- If L is regular, then $h(L)$ must be regular as well.
- However, we know $h(L)$ is **NOT** regular.
- Therefore, L is **NOT** regular. $\square$

Summary

1. Pumping Lemma for Regular Languages
   - Intuition
   - Pumping Lemma
   - Proof of Pumping Lemma
   - Application: Proving Languages are Not Regular

2. Examples
   - Example 1: $L = \{\texttt{a}^n\texttt{b}^n \mid n \geq 0\}$
   - Example 2: $L = \{ww^R \mid w \in \{\texttt{a}, \texttt{b}\}^*\}$
   - Example 3: $L = \{\texttt{a}^l\texttt{b}^m\texttt{c}^n \mid l + m \leq n\}$
   - Example 4: $L = \{\texttt{a}^{n^2} \mid n \geq 0\}$
   - Example 5: $L = \{\texttt{a}^n\texttt{b}^k\texttt{c}^{n+k} \mid n, k \geq 0\}$

Next Lecture

- Equivalence and Minimization of Finite Automata

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