Lecture 2 – Control Flow Graphs

SWS128: Software Analysis

Jihyeok Park

2026 Fall

Recall

- MiniPy: one `def`, no calls, lists as the only heap objects.
- Big-step semantics, and an interpreter for it.

- Today: **all executions at once**, without running any.

Contents

Contents

What and Why

- A **control flow graph** (CFG): **program points** joined by edges, one command per edge.
- The AST says how a program is *nested*. The CFG says where control can *go*.
- Every analysis in this course runs on it.

Definition

A **control flow graph** $G = (N, E, \texttt{entry}, \texttt{exit})$:

- N — **program points**.
- E — edges $p \xrightarrow{c} q$, each with one **command**:

$$c ::= x\texttt{:=}e \mid x\texttt{[}e\texttt{]:=}e \mid \texttt{pass} \mid \texttt{assume}(e)$$

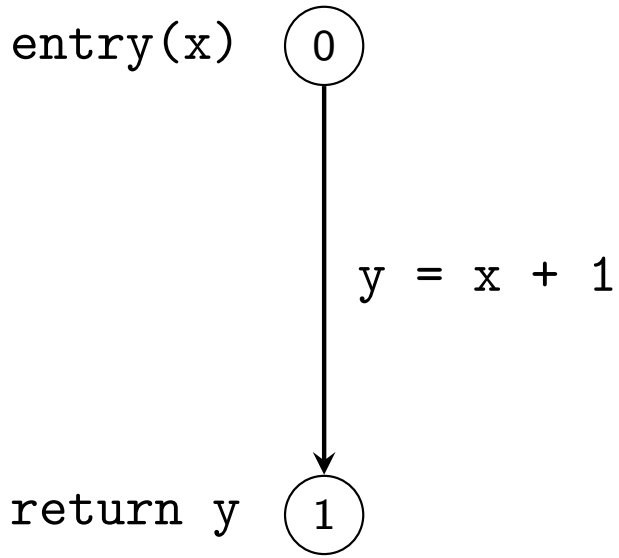
- $\texttt{entry}, \texttt{exit} \in N$ — parameters in, `return` out.

- $\texttt{assume}(e)$: can be taken only when e is true.

- A **path**: $\texttt{entry} \to p_1 \to \cdots \to p_k$. Every run follows some path.

A Straight Line

```
def f(x):
    y = x + 1
    return y
```



A Branch

```
def f(x):
    if 0 < x:
        y = 1
    else:
        y = 2
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```

A Branch

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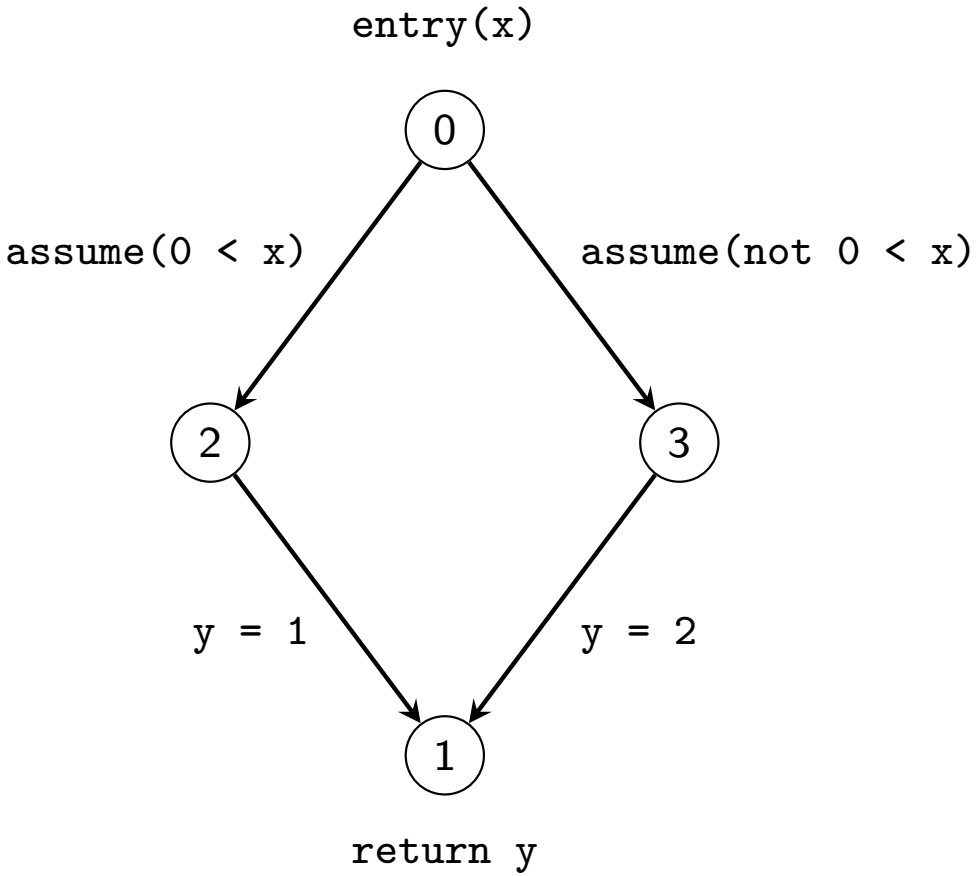
- A condition: two assume edges.



A Loop

```
def f(x):
    while x < 100:
        x = x * 2
    return x
```

A Loop

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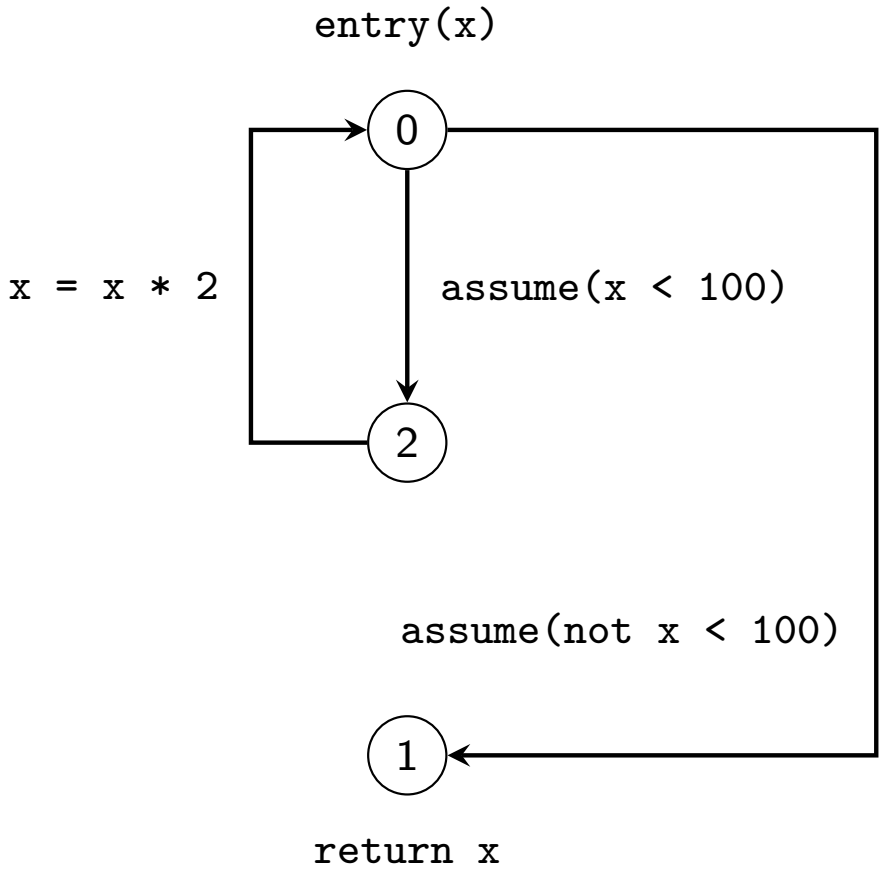
- The body comes **back**: a **loop head** and a **back edge**.



The Whole Program

```
def total(n):
    xs = [3, -1, 4]
    s = 0
    i = 0
    while i < n:
        if 0 <= xs[i]:
            s = s + xs[i]
        i = i + 1
    return s
```

The Whole Program

```
def total(n):
    xs = [3, -1, 4]
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```
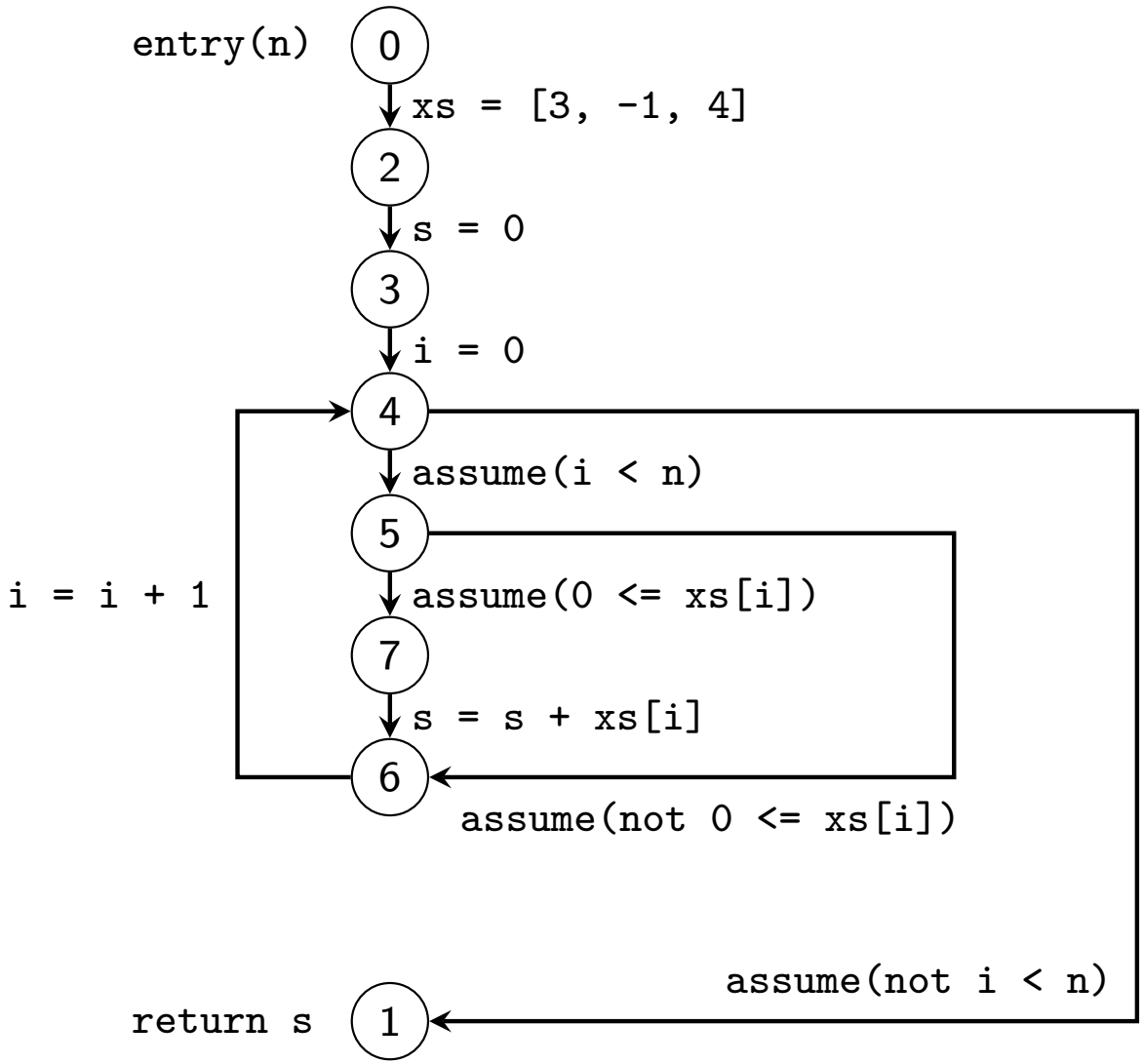

- Nesting is gone. Only flow remains.



Semantics

A step takes an edge and runs its command:

$$\langle p, \sigma \rangle \to \langle q, \sigma' \rangle \qquad \text{iff} \qquad p \xrightarrow{c} q \;\text{ and }\; \sigma \vdash c \Rightarrow \sigma'$$

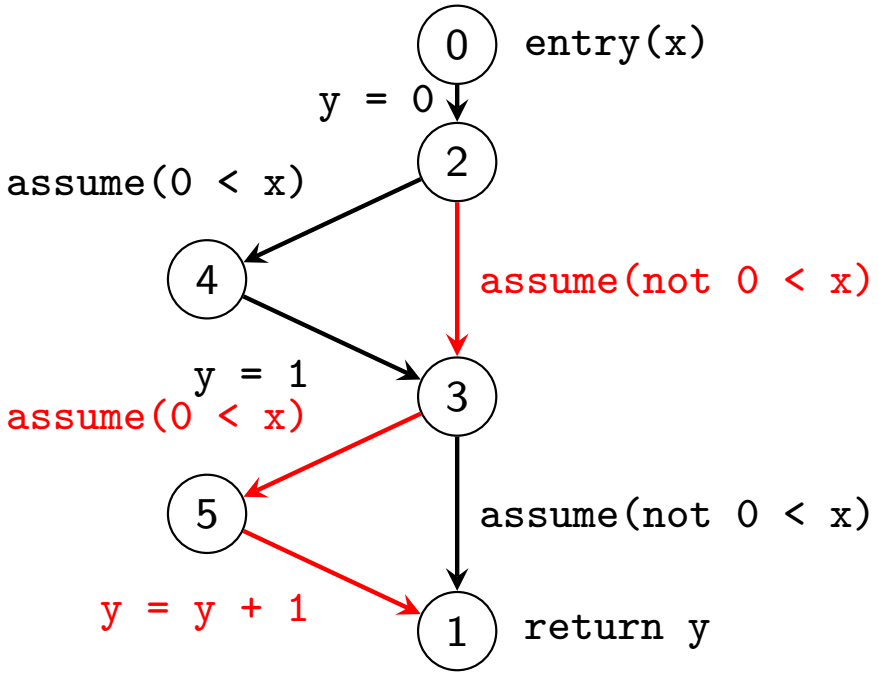
The new command:

$$\frac{\sigma \vdash e \Rightarrow \texttt{true}}{\sigma \vdash \texttt{assume}(e) \Rightarrow \sigma}$$

- A false `assume` has no rule: that edge cannot be taken.
- A run is a path on which **every `assume` held**.

Infeasible Paths

```
def f(x):
    y = 0
    if 0 < x:
        y = 1
    if 0 < x:
        y = y + 1
    return y
```



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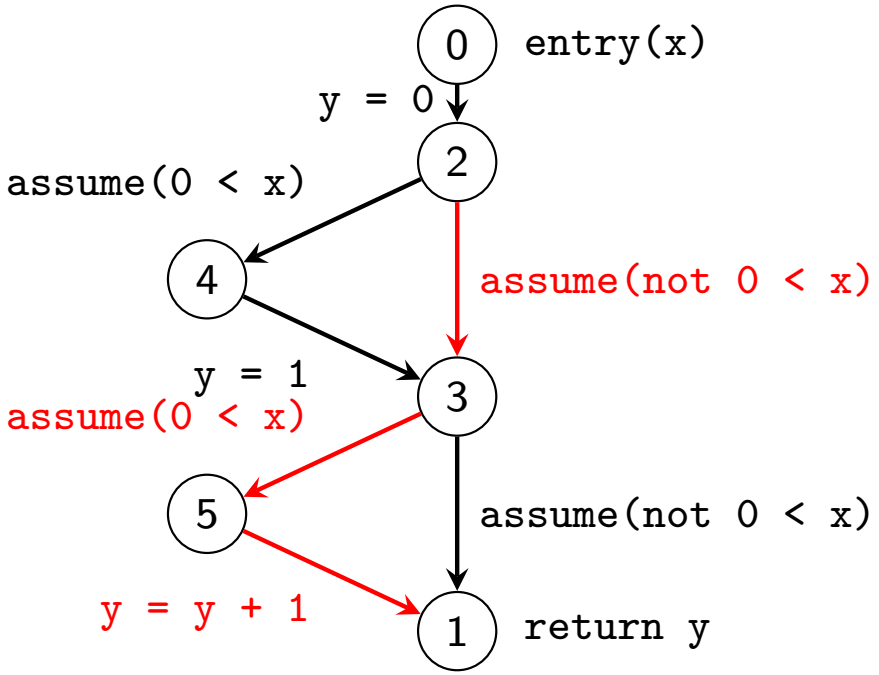

- The red path returns 1. No input does.

Infeasible Paths

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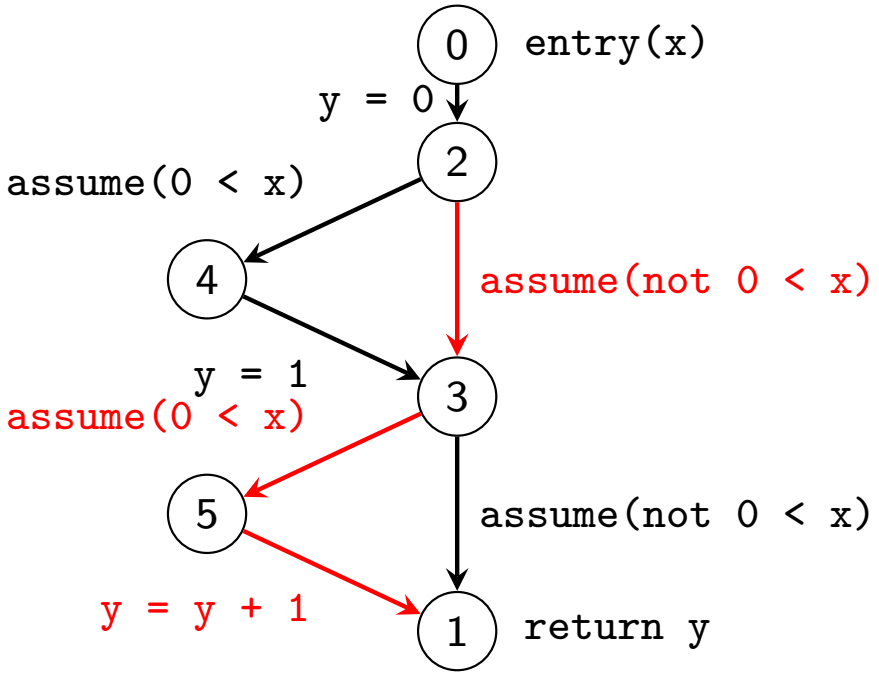


- The red path returns 1. No input does.
- **Over-approximation starts here.**

Contents

Translation by Cases

translate$(s, \text{src}, \text{dst})$: edges that run s from src to dst.

- `x = e`: one edge src $\xrightarrow{s}$ dst.
- $s_1 ; s_2$: a fresh point m; src $\to s_1 \to m \to s_2 \to$ dst.
- `if c: A else: B`: src $\xrightarrow{\texttt{assume(c)}} A \to$ dst and src $\xrightarrow{\texttt{assume(not c)}} B \to$ dst.
- `while c: A`: src $\xrightarrow{\texttt{assume(c)}} A \to$ src and src $\xrightarrow{\texttt{assume(not c)}}$ dst.
- The program: translate$(s^+, \texttt{entry}, \texttt{exit})$.

In Python – Edges

```python
@dataclass(frozen=True)
class Assume:
    cond: Expr

Cmd = Stmt | Assume

@dataclass(frozen=True)
class Edge:
    id: int
    src: int
    cmd: Cmd
    dst: int
```

- A point is just an `int`.

In Python – The Graph

```
@dataclass
class CFG:                                  # (defaults omitted)
    params: tuple[str, ...]                 # bound at the entry: the input
    ret: Expr                               # evaluated at the exit: the output
    edges: list[Edge]
    succ: dict[int, list[Edge]]
    pred: dict[int, list[Edge]]
    loop_heads: set[int]
    entry: int
    exit: int
```

In Python – The Program

```
def build(prog: Prog) -> CFG:
    g = CFG(prog.params, prog.ret)
    g.entry = g.add()
    g.exit = g.add()
    _block(g, prog.body, g.entry, g.exit)
    return g

def _block(g: CFG, body: Block, src: int, dst: int) -> None:
    """Edges for `body`, from point src to point dst."""
    if not body:
        g.edge(src, Pass(), dst)
        return
    for st in body[:-1]:
        mid = g.add()
        _stmt(g, st, src, mid)
        src = mid
    _stmt(g, body[-1], src, dst)
```

In Python – One Statement

```python
def _stmt(g: CFG, st: Stmt, src: int, dst: int) -> None:
    match st:
        case If(cond, then, els):
            _branch(g, cond, then, src, dst)
            _branch(g, UnOp("not", cond), els, src, dst)
        case While(cond, body):
            g.loop_heads.add(src)
            _branch(g, cond, body, src, src)
            g.edge(src, Assume(UnOp("not", cond)), dst)
        case _:                              # x = e | x[e] = e | pass
            g.edge(src, st, dst)

def _branch(g: CFG, cond: Expr, body: Block, src: int, dst: int) -> None:
    """src --assume(cond)--> body --> dst."""
    if not body:
        g.edge(src, Assume(cond), dst)
    else:
        mid = g.add()
        g.edge(src, Assume(cond), mid)
        _block(g, body, mid, dst)
```

Basic Blocks

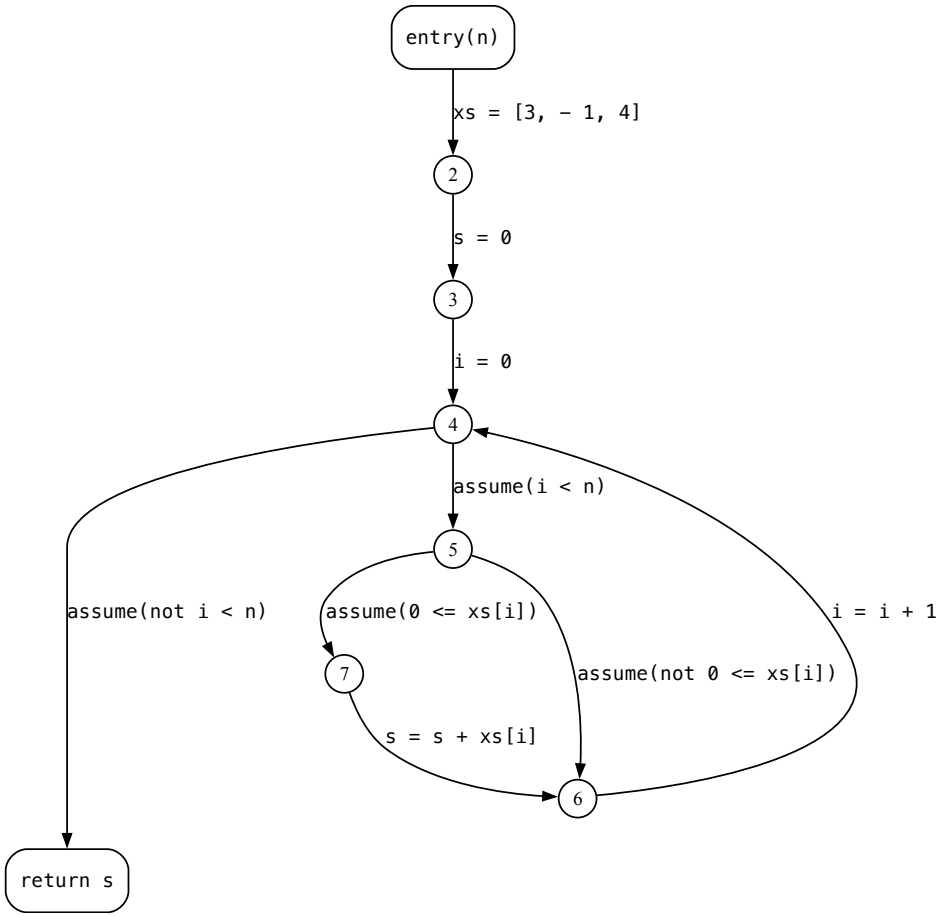
- Compilers put a straight-line sequence on one edge: a **basic block**. In `total`: `xs = [3, -1, 4]`, `s = 0`, `i = 0` form one block.

- We keep one command per edge. Simpler, and our programs are small.

Seeing the Graph

```
$ python3 cfg.py example.py | dot -Tpdf -o cfg.pdf
```

```
def total(n):
    xs = [3, -1, 4]
    s = 0
    i = 0
    while i < n:
        if 0 <= xs[i]:
            s = s + xs[i]
        i = i + 1
    return s
```



dot is Graphviz, optional: **graphviz.org/download**

Running It

```
>>> from syntax import parse
>>> from cfg import build, find
>>> g = build(parse(open("example.py").read()))

>>> len(g.nodes), len(g.edges)
(8, 9)

>>> g.params, g.entry, g.exit
(('n',), 0, 1)

>>> g.loop_heads
{4}

>>> [str(e) for e in g.succ[4]]
['assume(i < n)', 'assume(not i < n)']

>>> e = find(g, "s = s + xs[i]")
>>> e.src, e.dst
(7, 6)
```

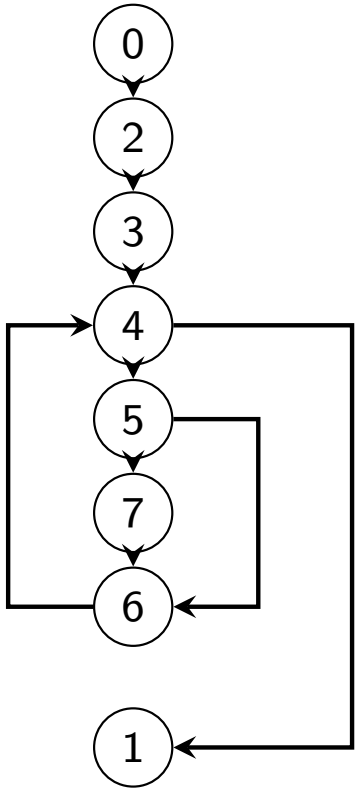
Contents

Dominators

d **dominates** p if **every** path from `entry` to p passes through d.

- Every point dominates itself. `entry` dominates everything.
- The **immediate dominator**: the closest strict dominator. They form a tree.
- Meaning: "d **must** have happened before p."

Dominators: Example



the CFG

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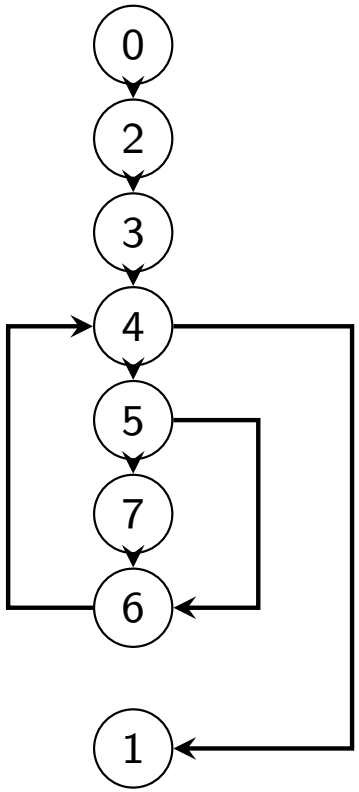
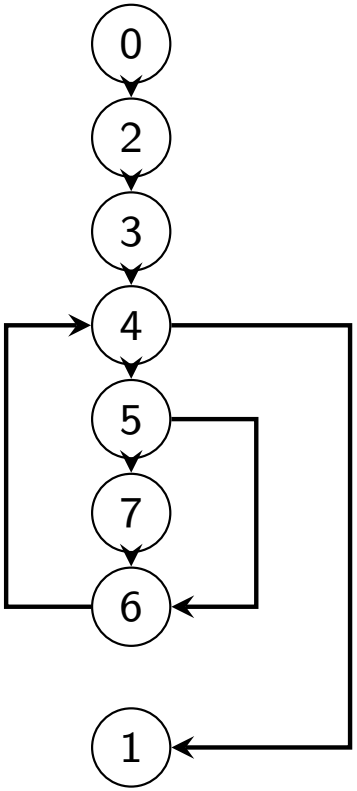


the CFG

p	$\mathsf{dom}(p)$
0	$\{0\}$
2	$\{0, 2\}$
3	$\{0, 2, 3\}$
4	$\{0, 2, 3, 4\}$
5	$\{0, 2, 3, 4, 5\}$
7	$\{0, 2, 3, 4, 5, 7\}$
6	$\{0, 2, 3, 4, 5, 6\}$
1	$\{0, 2, 3, 4, 1\}$

Dominators: Example



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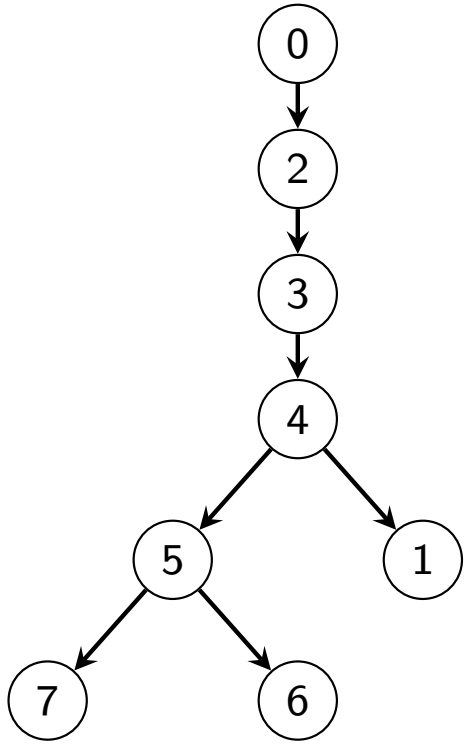

the dominator tree

A Recursive Definition

$$\mathsf{dom}(\texttt{entry}) = \{\texttt{entry}\} \qquad \mathsf{dom}(p) = \{p\} \cup \bigcap_{q \to p} \mathsf{dom}(q) \quad (p \neq \texttt{entry})$$

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- $\text{dom}(p)$ is defined by the $\text{dom}(q)$ of its predecessors — and around a loop, by itself. **Recursive.**
- So: start from a guess, and apply the definition again and again until it stops changing.

Iterating to the Fixpoint

One round applies the definition at every point:

$$F(D)(p) = \{p\} \cup \bigcap_{q \to p} D(q)$$

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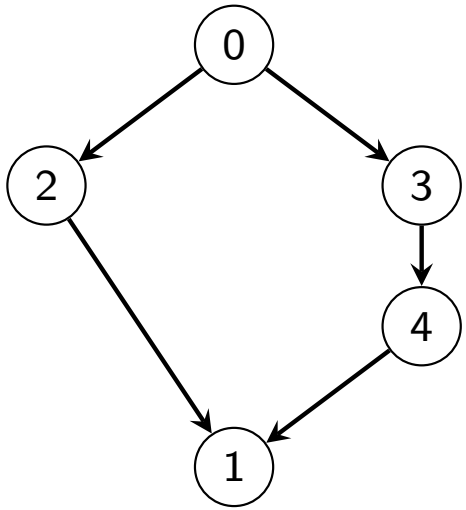
$$D_0(\texttt{entry}) = \{\texttt{entry}\}, \quad D_0(p) = N \ (p \neq \texttt{entry})$$

$$D_{i+1} = F(D_i) \qquad \text{stop when } D_{i+1} = D_i$$

- Start from the largest guess: a round only **removes**. Sets are finite, so it **stops**.
- Where it stops, $F(D) = D$: the definition holds at every point. A **fixpoint**.

Iterating: Example

```
def f(x):
    if 0 < x:
        y = 1
    else:
        y = 2
        y = y * 2
    return y
```



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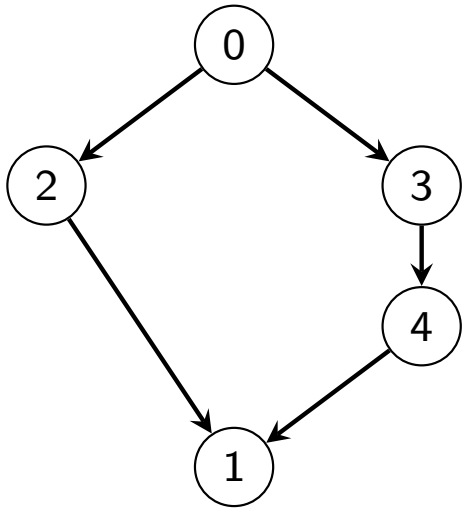

p	D_0	D_1	D_2	$D_3 = D_4$
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	N	$\{0, 2\}$	$\{0, 2\}$	$\{0, 2\}$
3	N	$\{0, 3\}$	$\{0, 3\}$	$\{0, 3\}$
4	N	N	$\{0, 3, 4\}$	$\{0, 3, 4\}$
1	N	N	$\{0, 1, 2\}$	$\{0, 1\}$

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def f(x):
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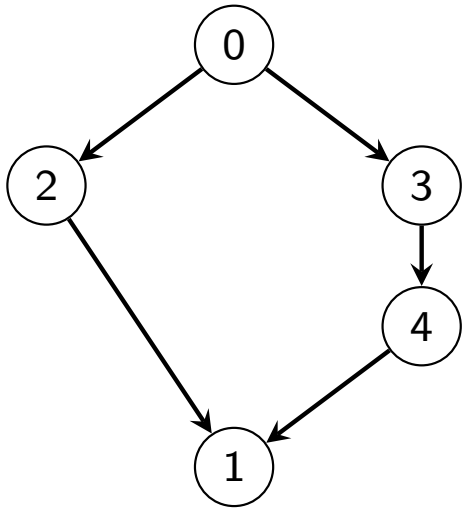

p	D_0	D_1	D_2	$D_3 = D_4$
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2	N	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$
3	N	$\{0,3\}$	$\{0,3\}$	$\{0,3\}$
4	N	N	$\{0,3,4\}$	$\{0,3,4\}$
1	N	N	$\{0,1,2\}$	$\{0,1\}$

- $\mathrm{dom}(1)$ shrinks **twice**: first the short branch arrives, then the long one.

The Worklist

Round by round recomputes *every* point. Only points whose predecessors changed can change:

1: $D(p) \leftarrow$ initial for all p
2: $W \leftarrow$ all points but `entry`
3: **while** $W \neq \emptyset$ **do**
4: pop p from W
5: new $\leftarrow F(D)(p)$
6: **if** new $\neq D(p)$ **then**
7: $D(p) \leftarrow$ new
8: push every successor of p onto W, unless already there

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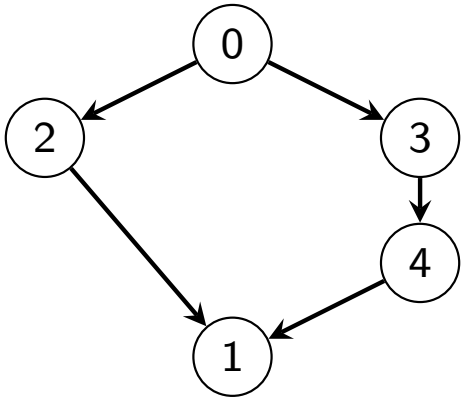
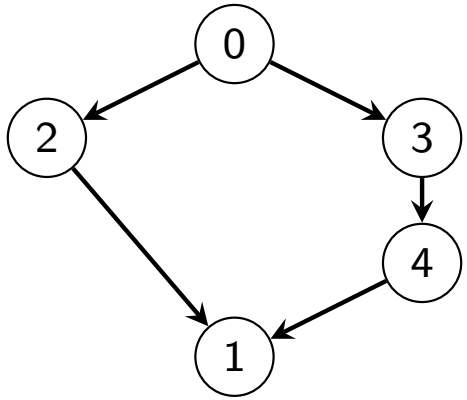
- Same fixpoint, less work. Empty worklist $\Rightarrow$ nothing left that can change.
- W can be a queue, a stack, or anything else: the order changes the work, not the answer.

The Worklist: Example



step	pop	dom(0)	dom(2)	dom(3)	dom(4)	dom(1)	push	W
0		$\{0\}$	N	N	N	N		$1, 2, 3, 4$
1	1	$\{0\}$	N	N	N	N		$2, 3, 4$
2	2	$\{0\}$	$\{0, 2\}$	N	N	N	1	$3, 4, 1$
3	3	$\{0\}$	$\{0, 2\}$	$\{0, 3\}$	N	N	4	$4, 1$
4	4	$\{0\}$	$\{0, 2\}$	$\{0, 3\}$	$\{0, 3, 4\}$	N	1	1
5	1	$\{0\}$	$\{0, 2\}$	$\{0, 3\}$	$\{0, 3, 4\}$	$\{0, 1\}$		$\emptyset$

The Worklist: Example



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2	2	$\{0\}$	$\{0, 2\}$	N	N	N	1	$3, 4, 1$
3	3	$\{0\}$	$\{0, 2\}$	$\{0, 3\}$	N	N	4	$4, 1$
4	4	$\{0\}$	$\{0, 2\}$	$\{0, 3\}$	$\{0, 3, 4\}$	N	1	1
5	1	$\{0\}$	$\{0, 2\}$	$\{0, 3\}$	$\{0, 3, 4\}$	$\{0, 1\}$		$\emptyset$

- Five pops instead of 4×4. Steps 3 and 4 push a point already in W.

In Python – The Worklist

PLRG

```
class Worklist:
    """A queue of points that holds each point at most once."""

    def __init__(self, points):
        self.queue, self.members = deque(), set()
        for p in points:
            self.push(p)

    def push(self, p: int) -> None:
        if p not in self.members:
            self.queue.append(p)
            self.members.add(p)

    def pop(self) -> int:
        p = self.queue.popleft()
        self.members.remove(p)
        return p

    def __bool__(self) -> bool:
        return bool(self.queue)
```

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    def __bool__(self) -> bool:
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```

- A queue plus a set: pushing a point already inside does nothing.

In Python – Dominators

```python
def dominators(g: CFG) -> dict[int, set[int]]:
    """dom(p) = {p} union (intersection of dom(q) for every edge q -> p)."""
    # initial: the entry is its own dominator; every other point starts full
    dom = {p: (set(g.nodes) if p != g.entry else {g.entry}) for p in g}
    # worklist: every point but the entry, whose equation is already solved
    work = Worklist(p for p in g if p != g.entry)
    while work:
        p = work.pop()
        # transfer: the equation for p
        new = {p} | set.intersection(*(dom[e.src] for e in g.pred[p]))
        if new != dom[p]:
            dom[p] = new
            for e in g.succ[p]:
                if e.dst != g.entry:
                    work.push(e.dst)
    return dom
```

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        if new != dom[p]:
            dom[p] = new
            for e in g.succ[p]:
                if e.dst != g.entry:
                    work.push(e.dst)
    return dom
```

- Three things: an **initial** value, a **worklist**, a **transfer** function.

The Recipe

PLRG

An analysis is three things:

- **Initial**: the starting value at every point.
- **Transfer** F: how a point's value is recomputed from its neighbours.
- **Worklist**: the points still to recompute. Pop one, apply F, push its neighbours if it changed. Empty $\Rightarrow$ fixpoint.

- **Every analysis in this course fixes these three and runs the same loop.**

SSA Form

Static Single Assignment: one operator per line, every variable assigned once.

```
x = a + b * c
x = x + 1
```

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```
t1 = b * c
x1 = a + t1
x2 = x1 + 1
```

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t1 = b * c
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x2 = x1 + 1
```

- Every use names **exactly one** assignment.

SSA Form: A Branch

```
def f(x):
    if 0 < x:
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    else:
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    return y
```

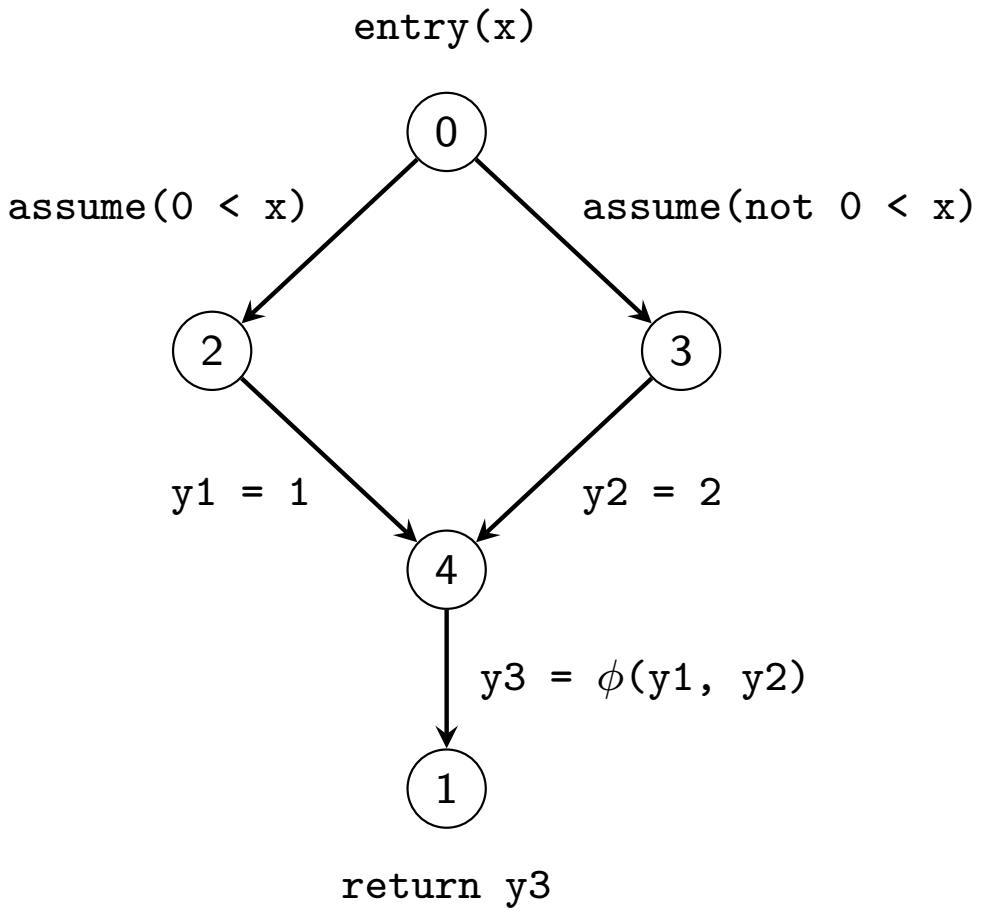

- At 4, y could be y1 or y2. Every use must name **one** assignment.
- So the join is followed by a new one: ϕ picks the side control came from.

SSA Form: A Loop

```
def f(n):
    i = 0
    while i < n:
        i = i + 1
    return i
```

SSA Form: A Loop

```
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    i = 0
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        i = i + 1
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```
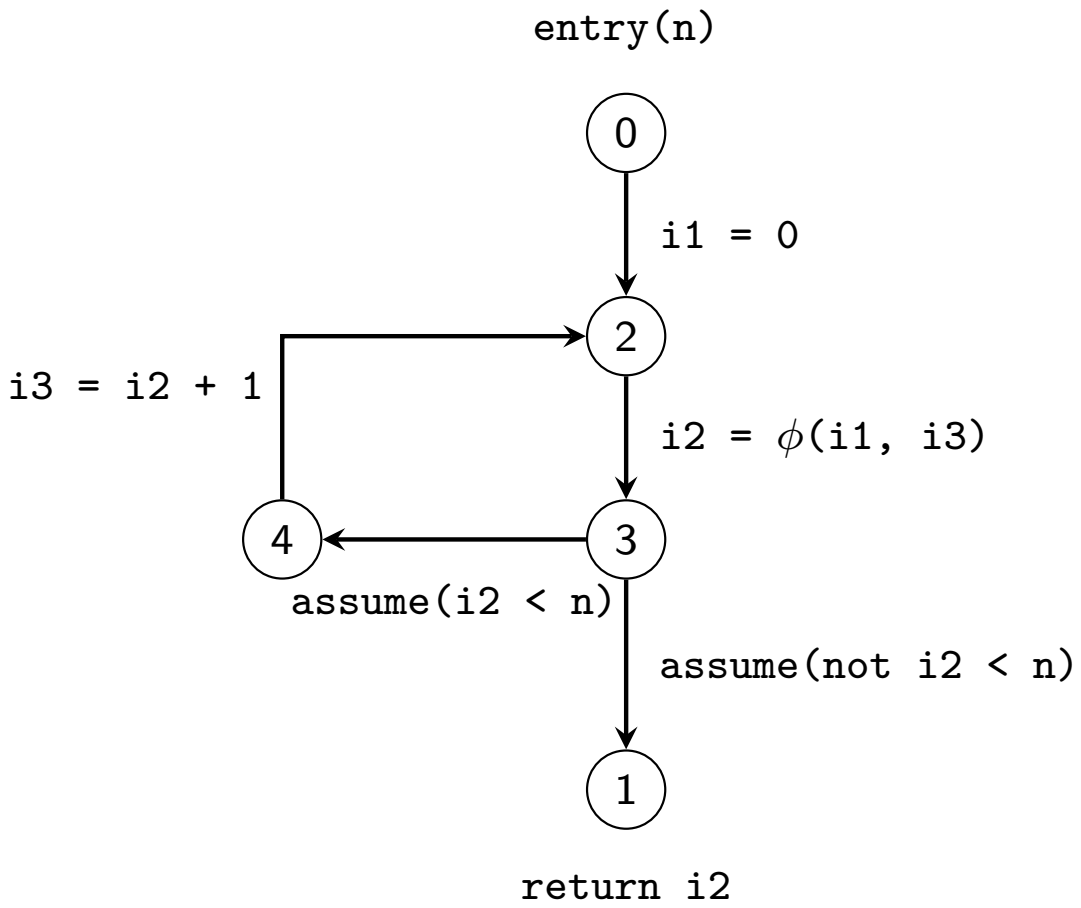

- The loop head 2 is a join too: from 0, and from the back edge.
- Its ϕ edge is taken every time control reaches 2.



Where Do the ϕs Go?

Inside the region d dominates, a definition leaving d is the only one. The **dominance frontier** is where that region ends:

$$\mathsf{DF}(d) = \{\, p \mid \exists\, q \to p.\;\; d \in \mathsf{dom}(q) \;\;\text{and}\;\; d \notin \mathsf{dom}(p) \setminus \{p\} \,\}$$

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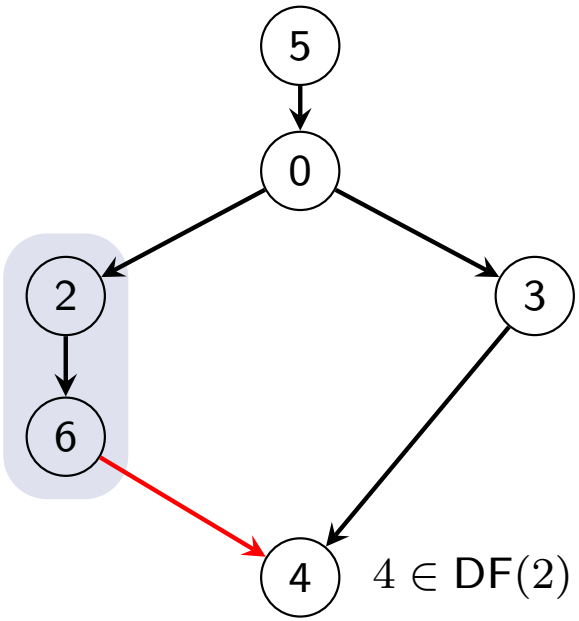

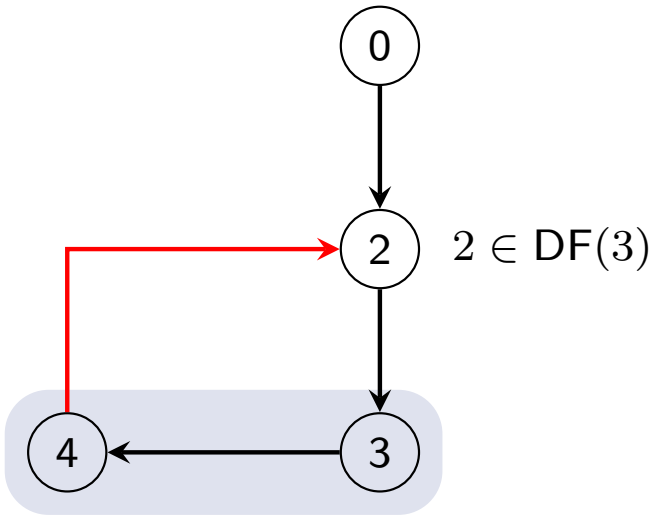

- The red edge leaves the region: its target is reached *through* d and *around* d. A ϕ goes there.
- LLVM works in SSA. We do not.

Summary

PLRG

1. Control Flow Graphs
    - Definition
    - Examples
    - Semantics
2. Construction
    - Translation
    - Basic Blocks
    - Seeing the Graph
    - Running It
3. Dominators and SSA
    - Dominators
    - Fixpoint Iteration
    - SSA Form

Next Lecture

- Dataflow Analysis

Jihyeok Park
jihyeok_park@korea.ac.kr
https://plrg.korea.ac.kr